MATHEMATICS
APR - 7 1941

Mathematical Reviews

Vol. 2, No. 4 April, 1941 pp. 113–144

TABLE OF CONTENTS

History 113

Algebra 115
Equations, polynomials 116
Linear algebra 118
Abstract algebra 120

Theory of groups 124

Analysis 128
Theory of sets, theory of functions of real variables 128
Functional equations and operational calculus 132

Geometry 135
Algebraic geometry 136

Mathematical physics 138

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Published monthly by

THE AMERICAN MATHEMATICAL SOCIETY, Prince and Lemon Streets, Lancaster, Pennsylvania

Sponsored by

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THE MATHEMATICAL ASSOCIATION OF AMERICA
ACADEMIA NACIONAL DE CIENCIAS EXACTAS, FISICAS Y NATURALES DE LIMA
HET WISKUNDIG GENOOTSCHAP TE AMSTERDAM
THE LONDON MATHEMATICAL SOCIETY

Editorial Office

MATHEMATICAL REVIEWS, Brown University, Providence, R. I.

Subscriptions: Price $13 per year ($6.50 per year to members of sponsoring societies). Checks should be made payable to MATHEMATICAL REVIEWS. Subscriptions should be addressed to MATHEMATICAL REVIEWS, Lancaster, Pennsylvania, or Brown University, Providence, Rhode Island.

This publication was made possible in part by funds granted by the Carnegie Corporation of New York, the Rockefeller Foundation, and the American Philosophical Society held at Philadelphia for Promoting Useful Knowledge. These organizations are not, however, the authors, owners, publishers, or proprietors of this publication, and are not to be understood as approving by virtue of their grants any of the statements made or views expressed therein.

Entered as second-class matter February 3, 1940 at the post office at Lancaster, Pennsylvania, under the act of March 3, 1879. Accepted for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in paragraph 4, section 538, P. L. and R. authorized November 9, 1940.

Mathematical Reviews

Vol. 2, No. 4 APRIL, 1941 Pages 113-144

HISTORY

Hofmann, Jos. E. Über Ziele und Wege mathematikgeschichtlicher Forschung. Deutsche Math. 5, 150–157 (1940). [MF 2816]

***Bell, E. T. The Development of Mathematics.** McGraw-Hill Book Company, Inc., New York, 1940. xiii+583 pp. $4.50.

Bell's new book is a source book of information on a vast field of modern and past mathematical research. This is revealed by the titles of chapters such as "Extensions of Number," "Toward Mathematical Structure," "Arithmetic Generalized," "Emergence of Structural Analysis," "Cardinal and Ordinal to 1902," "From Intuition to Absolute Rigor, 1700–1900," "Rational Arithmetic after Fermat," "From Mechanics to Generalized Variables," "Differential and Difference Equations," "Invariance" and as last "Uncertainties and Probabilities." Authors and teachers working on a particular subject will do well to consult Bell's book in order to obtain a broad survey of its outstanding results and the main authors up to the present time. The selection of the material has been performed, not so much with an eye on impartial completeness as Cajori's "History of Mathematics" (which, besides, stopped at 1919) but, in the words of the author "after consultation with numerous professionals who knew from hard personal experience what mathematical invention means. On their advice, only main trends of the past thousand years are considered, and these are presented only through typical major episodes in each." The experience of the author as a creative mathematician, a teacher and interested colleague has made it possible to place lively comments, pithy summaries and challenging outlooks between an otherwise factual survey of achievements. These surveys have unequal merits, excelling in arithmetic and algebra and in related fields, and losing their completeness somewhat in regions less professionally familiar to the author. The central importance of Laplace in the theory of probabilities can hardly be grasped from the information in Bell's book that his book is a "masterpiece" and "distinguished by a free use of mathematical analysis." Geometry is rather stepmotherly treated; we miss outstanding names like Bianchi, Blaschke, Segre; we find that Cartan only comes in through his thesis of 1894 and that Study's merits consist in "extending the method of differential operators to any quantic."

The first hundred odd pages are devoted to mathematics before 1637 and bring a bird's eye view for which "the works of professional historians of mathematics have been used for some matters on which the historians are in approximate agreement among themselves." The summary will be quite useful for students who like to see the main landmarks, though it treats many points with more charm than mastery of source material. We do not get the impression that the work of modern historians of mathematics, as Junge, Frank, Dyksterhuis, Loria, has been of much influence on the narrative. Forty pages are taken by "The Beginning of Modern Mathematics, 1637–1687," a good account of the period, though we meet again a certain refusal to evaluate the real greatness of some individuals, here notably Leibniz. This book, we believe, is a welcome supplement to Cajori's "History," which it surpasses in its grasp of many fundamental problems, but it suffers if compared to Klein's monumental history, which, though only a rump, still sets the example for all future histories of modern mathematics. *D. J. Struik* (Cambridge, Mass.).

***Coolidge, Julian Lowell. A History of Geometrical Methods.** Oxford University Press, New York, 1940. xviii+451 pp. $10.00.

Michel Charles' classical "Aperçu historique" of 1837 inspired Coolidge, after a hundred years, to write "a new book dealing with the history of the methods which men have employed in dealing with geometrical questions." There are four main divisions of geometry, synthetic geometry, algebraic geometry, differential geometry and topology. The author decided to omit topology as being somewhat outside his own interests and as being still too much in a state of change. The result is a thorough and exceedingly well written symposium on the three older types of geometry. Book I, Synthetic Geometry, has six chapters, on the beginnings of geometry, Greek mathematics, later elementary geometry, the non-euclidean geometries, projective and descriptive geometry. Book II, Algebraic Geometry, deals in seven chapters with beginnings, extension of the system of linear coordinates, other systems of point coordinates, enumerative geometry, birational geometry, higher spaces and higher space elements and geometrical transformations. Book III, Differential Geometry, has five chapters, on early writers, intrinsic geometry, Gauss and the classical theory of surfaces, projective differential geometry and absolute differential geometry. The book ends with an excellent list of books and papers quoted.

This is the book to be consulted by everyone who wants to know what we might call modern classical geometry and its history. It is not only a summary or description of results, but a systematic development of the material with proofs and corollaries, a kind of multiform textbook, from which the student can not only get an orientation, but which he can use as a basis for his studies. A great number of special topics are briefly or amply discussed, from the geometry of the spider's web to modern criticism of enumerative geometry, Douglas' work on the Plateau problem, quaternions and some tensor analysis. Anecdotes and personal remarks enliven the argumentation. The author, in his attempt to present every field as clearly as possible on its own merits, had on occasion no sufficient opportunity to establish relationships underlying different theories. Tensors, treated in a last special chapter, are in this book more divorced from vectors, homogeneous coordinates, Clebsch Aronhold symbolism, Grassmann magnitudes and fundamental forms of the surface than they really are. The groups

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underlying the Ausdehnungslehre, vectors, quaternions, biquaternions and screws, which give each theory its distinctive place, are not particularly indicated despite the prominent place accorded to Klein's group concepts. It is a pity that C. L. E. Moore's work is not mentioned though it is entirely in the good Segre and Ricci tradition, and though Moore was one of the pioneers of the tensor calculus outside of Italy. *D. J. Struik* (Cambridge, Mass.).

*Tropfke, Johannes. **Geschichte der Elementar-Mathematik in systematischer Darstellung mit besonderer Berücksichtigung der Fachwörter. Bd. 4. Ebene Geometrie.** 3rd ed. de Gruyter, Berlin, 1940. 316 pp. 13 RM.

*De Vries, Hk. **Historische Studiën. Deel III.** P. Noordhoff, Groningen, 1940. 261 pp. f. 3.75.
Part II appeared in 1934.

Thaer, Clemens. Die Würfelverdoppelung des Apollonios. Deutsche Math. **5**, 241–243 (1940). [MF 3119]
The Greek tradition leaves some doubt as to how the solution of the Delian problem, ascribed to Apollonius, can be connected with conic sections, mentioned by Pappus as used by Apollonius. The author shows that such a method actually occurs in the "little edition" of Euclid by al-Ṭûsî (1248) as a lemma to a "theorem of Apollonius."
O. Neugebauer (Providence, R. I.).

Müller, Conrad. Volumen und Oberfläche der Kugel bei Âryabhaṭa I. Deutsche Math. **5**, 244–255 (1940). [MF 3120]
The author discusses stanzas 6 and 7 of section 2, the so-called gaṇitapâda of the Âryabhaṭîyam, written by Âryabhaṭa I. Stanza 6 deals with the area of a triangle and the volume of a pyramid, and stanza 7 deals with the area of a circle and the volume of a sphere. In the two modern translations of the Âryabhaṭîyam by W. E. Clark and P. C. Sengupta these two stanzas have been translated in such a way that the resulting formula for the volume of a pyramid is $\frac{1}{2}hA$, and the formula for the volume of a sphere is $\pi\sqrt{\pi r^3}$. Conrad Müller now gives a translation of these two stanzas by which he obtains correct formulas. The key to the correct understanding of stanza 6 is a better translation of the words "phala śarîram." Clark seems to have translated phala śarîram as "area," translating only phala and omitting śarîram. Müller translates phala śarîram as "a solid obtained from the area (of the triangle)"; that is, a special pyramid which, when unfolded, is a triangle. From this point of view the stanza gives formulas for the lateral area and for the volume of a regular triangular pyramid. The word which formerly was translated into "a solid which has six edges (pyramid)" is now translated into "a prism equivalent to six (of the mentioned) pyramids." It seems to me doubtful that "ghana," which in stanza 6 was translated as "6 pyramids," can be translated as "6 spheres" in stanza 7. *O. Schmidt.*

Weiss, E. A. Die kennzeichnende Eigenschaft des Österreichischen Fasses. Deutsche Math. **5**, 262–265 (1940). [MF 3121]
Remark concerning Kepler's "Stereometria doliorum."
O. Neugebauer (Providence, R. I.).

Vogel, K. Zur Geschichte der linearen Gleichungen mit mehreren Unbekannten. Deutsche Math. **5**, 217–240 (1940). [MF 3118]
It is shown that Leonardo of Pisa in his "Liber abbaci" develops rules for solving systems of linear equations, attempting both proof and generalization to an arbitrary number of equations. Even negative quantities ("debts") are taken into consideration. Relations to Diophantus are emphasized. *O. Neugebauer* (Providence, R. I.).

Frajese, Attilio. L'algebra geometrica in Leonardo Pisano. Boll. Un. Mat. Ital. (2) **2**, 363–365 (1940). [MF 2990]

Smith, David Eugene. Francisco Vieta, 1540–1603. Bol. Mat. **13**, 221–223 (1940). (Spanish) [MF 3149]

Agostini, Amedeo. I baricentri di gravi non omogenei e la formola generale per il loro calcolo determinati da Bonaventura Cavalieri. Boll. Un. Mat. Ital. (2) **2**, 147–171 (1940). [MF 2971]

Geppert, Harald. Wie Gauss zur elliptischen Modulfunktion kam. Deutsche Math. **5**, 158–175 (1940). [MF 2817]
The paper is an address delivered before the second German Camp of Mathematicians and gives an account of the investigations of Gauss concerning the elliptic modular function. Gauss did not publish anything about these researches which anticipate a considerable part of the later work of Abel and Jacobi. The results found in his notebooks after his death were edited and discussed in the Collected Works of Gauss by Schering, Klein, Fricke and Schlesinger. The author makes some further suggestions on the ideas which led Gauss to his discoveries.
C. L. Siegel (Princeton, N. J.).

Bernstein, S. N. The Petersburg school of the theory of probability. Leningrad State Univ. Annals [Uchenye Zapiski] Math. Ser. **10**, 3–11 (1940). (Russian) [MF 3299]

Gontcharoff, V. et Kolmogoroff, A. Le soixantenaire de S. Bernstein. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **4**, 249–260 (1940). (Russian) [MF 3321]
A biographical sketch with a list of publications.
J. D. Tamarkin (Providence, R. I.).

*Lange, H. O. und Neugebauer, O. **Papyrus Carlsberg No. I. Ein hieratisch-demotischer kosmologischer Text.** Danske Vid. Selsk. Hist.-Filol. Skrifter **1**, no. 2, 92 pp. 7 plates and 31 tables. 35 Danish Kroner.

Pannekoek, A. Some remarks on the moon's diameter and the eclipse tables in Babylonian astronomy. Eudemus **1**, 1–14 (1941). [MF 3807]
The author gives another interpretation of certain parameters in the Babylonian method for calculating eclipses, which were considered to be the diameter of the moon's disc and the measure of the magnitude of an eclipse by the reviewer in Quellen und Studien zur Geschichte der Mathematik (B) **4**, 193–346 (1937). The method of predicting eclipses is discussed. *O. Neugebauer* (Providence, R. I.).

van der Waerden, B. L. Die Voraussage von Finsternissen bei den Babyloniern. Ber. Verh. Sächs. Akad. Wiss. Leipzig **92**, 107–114 (1940). [MF 3132]
Investigations concerning the period-relations which may

explain the prediction of moon eclipses beginning with the 7th century B.C. As results there appear the relation 47 synodic months=51 draconitic months in the 7th and 6th centuries, then the "saros" (223 s.m.=242 d.m.), and finally the very accurate relations of "system II" and "system I."

O. Neugebauer (Providence, R. I.).

***Claudius Ptolemaeus. Opera quae exstant omnia. Vol. 3, part 1.** Ἀποτελεσματικά. Edited by F. Boll and E. Boer. B. G. Teubner, Leipzig, 1940. xviii+213 pp.

More than 350 years have passed since the second and last printed edition of this work of Ptolemy (usually quoted as the "Tetrabiblos") which exercised an influence on the medieval world comparable with that of Aristotle's writings. A modern edition is therefore of the highest interest for every scholar investigating the history of ancient and medieval astronomy. The present edition is based on 35 manuscripts, belonging to four different families, but all derived from one archetypus. It was necessary to postpone the publication of the indices to the second part of vol. III.

O. Neugebauer (Providence, R. I.).

***Coppernicus, Nicolaus. Über die Kreisbewegungen der Weltkörper.** Akademische Verlagsgesellschaft M. B. H., Leipzig, 1940. xvi+363 pp. text+66 pp. notes. RM 18.60.

Unaltered reprinting of the Thorn edition of 1879; translated and annotated by C. L. Menzzer.

Peters, Theodor. Jo. Kepleri Harmonices Mundi Liber I. Ein Beitrag zur Geschichte der Mathematik. Schr. Math. Inst. u. Inst. Angew. Math. Univ. Berlin **5**, 1–129 (1940). [MF 2780]

Kepler's point of view concerning problems of constructibility and irrationals in the sense of Euclid are discussed, mainly from a philosophical point of view.

O. Neugebauer (Providence, R. I.).

Archibald, R. C. Obituary: Thomas Little Heath. Math. Gaz. **24**, 234–237 (1940). [MF 3359]

Thompson, D'Arcy Wentworth. Obituary: Sir Thomas Heath, K.C.B., F.R.S., F.B.A. Nature **145**, 578–579 (1940). [MF 2958]

Thompson, D'Arcy W. Obituary: Sir Thomas Little Heath. 1861–1940. Obit. Notices Roy. Soc. London **1940**, 409–426 (1940). [MF 3806]

Obituary: Prof. Dr. Jan de Vries. Mathematica, Zutphen. A. **9**, 1 (1940). [MF 3528]

Obituary: Andrei Dmitrievič Arkhangelski. Vestnik Akad. Nauk SSSR **1940**, no. 8–9, 78–85 (1940). [MF 3697]

Tschebotareff, N. G. Obituary: D. A. Grave (1883–1939). Memorial volume dedicated to D. A. Grave [Sbornik posvjaščenii pamjati D. A. Grave], Moscow, 1940, pp. 1–14. **List of publications,** pp. 320–326. (Russian) [MF 3497]

Obituary: D. Grawe. Acad. Sci. RSS Ukraine. Rec. Trav. [Zbirnik Prace] Inst. Math. **1940**, no. 4, 1–6 (1940). (Ukrainian and Russian) [MF 3229]

Obituary: Dimitrii Aleksandrovič Grave. Rec. Math. [Mat. Sbornik] N.S. **7 (49)**, no. 2, i–ii (1940). (Russian) [MF 2788]

Born, M. Obituary: Prof. Otto Toeplitz. Nature **145**, 617 (1940). [MF 2959]

Hofreiter, Nikolaus. Nachruf auf Philipp Furtwängler. Monatsh. Math. Phys. **49**, 219–227 (1940). [MF 3088]

Obituary: Prof. Ludwig Hopf. Nature **145**, 379–380 (1940). [MF 2957]

Milne, E. A. Obituary: Prof. A. E. H. Love, F.R.S. Nature **146**, 393–394 (1940). [MF 2955]

Ferrar, W. L. Obituary: J. Hodgkinson. J. London Math. Soc. **15**, 236–240 (1940). [MF 3142]

Phillips, H. B. Obituary: Frank Morley (1860–1937). Proc. Amer. Acad. Arts Sci. **73**, 138–139 (1939). [MF 3007]

Dingle, Herbert. Obituary: Alfred Fowler, 1868–1940. Observatory **63**, 262–267 (1940). [MF 3087]

Gheorghiu, Gh. Th. Obituary: George Țițeica. Gaz. Mat. **45**, 169–173, 281–288, 339–344, 396–405 (1939). [To be continued] [MF 905]

ALGEBRA

Aigner, Alexander. Mathematische Behandlung des Einsiedlerspieles in der Ebene und im Raume. Deutsche Math. **5**, 12–36 (1940). [MF 2135]

Narasimha Murti, V. On a problem of arrangements. I. J. Indian Math. Soc. (N.S.) **4**, 39–43 (1940). [MF 2034]

The problem referred to is the following: $2n+1$ persons are invited to dinner on n different days. Is it possible to arrange them at a circular table in such a way that no person has the same neighbor on different days? This question is answered in the affirmative for all values of n by exhibiting a series of n such arrangements. The notation and discussion of the problem is unnecessarily complicated. By making a simple change in notation the solution of the problem is essentially as follows: Denote the $2n+1$ persons by $\infty, 0, 1, 2, \cdots, 2n-1$. Let ∞ occupy the same seat every day and let the remaining $2n$ seats be filled so that the kth seat ($1 \leqq k \leqq 2n$) on the rth day ($1 \leqq r \leqq n$) is occupied by the person whose number is congruent to $(-1)^k[k/2]+r-1$ modulo $2n$, where $[x]$ denotes, as usual, the greatest integer not exceeding x. With this assignment of seats the conditions of the problem are met.

D. H. Lehmer.

Levi, F. W. Remarks on Mr. V. Narasimha Murthi's paper: On a problem of arrangements (I). J. Indian Math. Soc. (N.S.) **4**, 45–46 (1940). [MF 2690]

For the paper referred to in the title cf. the preceding review. The author points out that the assertion that the dinner party problem has a solution for every n is equivalent to the statement that every complete graph with $2n+1$

vertices is a product (in the sense of superposition) of n simple circuits of $2n+1$ sides each. After discussing isomorphisms of such products the question is raised as to the number of non-isomorphic solutions for a particular n. This number is known to be at least two whenever $2n+1$ is a prime greater than 5. *D. H. Lehmer* (Berkeley, Calif.).

Barral Souto, J. Sums of products formed by combinations of given numbers. Bol. Mat. 13, 187–204 (1940). (Spanish) [MF 2867]

The author introduces the symbol

$$\binom{a_1, a_2, \cdots, a_n}{m}$$

for the combinatorial sum $\sum a_1 a_2 \cdots a_m$ of n elements, and shows its advantage by establishing easily a number of familiar combinatory formulas. *O. Szász.*

Megyesi, István. Lexikographische Ordnung von kombinatorischen Komplexionen. Mat. Fiz. Lapok 47, 178–179 (1940). (Hungarian. German summary) [MF 3713]

Es werden Formeln angegeben für die Bestimmung der Ordnungsnummer einer gegebenen Permutation, Kombination und Variation in der lexikographischen Ordnung der betreffenden Komplexionen. *Author's summary.*

Equations, Polynomials

Furtwängler, Ph. Über die Newtonschen Potenzsummenformeln. Monatsh. Math. Phys. 49, 194–196 (1940). [MF 2994]

Simple proof of a classical theorem in the elementary theory of equations. In the equation after (8), n is misprinted for g in several places. This is confusing since the author failed to state that, in equation (I''), s_0 must be replaced by g or $(n+k)$. *P. Franklin.*

Loria, Gino. W. von Tschirnhaus and algebraic equations with roots in arithmetical progression. Proc. Edinburgh Math. Soc. (2) 6, 181–184 (1940). [MF 2853]

Tschirnhaus gave without proof formulae for expressing the roots of an equation under the assumption mentioned in the title concerning the terms of two of the coefficients and for degrees from 2 to 9. The author now develops the corresponding relations for any degree n.

O. Neugebauer (Providence, R. I.).

Kennedy, E. C. Bounds for the roots of a trinomial equation. Amer. Math. Monthly 47, 468–470 (1940). [MF 2829]

Sergescu, P. Généralisations des limites de J. J. Bret. Acad. Roum. Bull. Sect. Sci. 22, 460–465 (1940). [MF 3202]

The paper deals with history and extension of classical elementary theorems on bounds for positive roots of equations with real coefficients. The following three theorems are cited from J. J. Bret [Ann. Math. Pures Appl. 6, 112–122 (1815)]: Given $f(x)=\sum_0^n \pm a_m x^{n-m}=0$, $a_m \geqq 0$, with real coefficients of which exactly μ, namely $-a_i, -a_j, \cdots, -a_s, -a_t$, $i<j<\cdots<s<t$, are negative. Let $a_m'=a_m$ for $m \neq i, j, \cdots, s, t$; $a_m'=0$ for $i, j, \cdots, s, t$. Let

$$b_1=a_i/(a_0'+a_1'+\cdots+a_{i-1}'),$$
$$b_2=a_j/(a_0'+a_1'+\cdots+a_{j-1}'), \cdots,$$
$$b_\mu=a_t/(a_0'+a_1'+\cdots+a_{t-1}').$$

Let $c_1=(a_i/a_0)^{1/i}$, $c_2=(a_j/a_0)^{1/(j-1)}$, $\cdots$, $c_\mu=(a_t/a_0)^{1/(t-\mu+1)}$.

Let

$$d_1=n-1+a_i/(n^i a_0'+n^{i-1}a_1'+\cdots+a_{n-1}'), \cdots,$$
$$d_\mu=n-1+a_t/(n^{t-\mu+1}a_0'+n^{t-\mu}a_1'+\cdots+a_{t-1}').$$

Then

$$\max(b_1+1, \cdots, b_\mu+1), \quad \max(c_1+1, \cdots, c_\mu+1), \quad \max(d_1+1, \cdots, d_\mu+1)$$

each represent an upper bound for the positive roots of $f(x)=0$. Extensions of these theorems are given.

A. J. Kempner (Boulder, Colo.).

Sergescu, Pierre. Sur les limites de J.-J. Bret. C. R. Acad. Sci. Paris 210, 652–654 (1940). [MF 3039]

Montel, Paul. Observations sur la communication précédente. C. R. Acad. Sci. Paris 210, 654–655 (1940). [MF 3039]

Let $f(x)=a_0x^m+a_1x^{m-1}+\cdots+a_m$ $(a_0>0)$ be a real polynomial in which only the coefficients a_j with $j=n_1, n_2, \cdots, n_k$ $(0<n_1<n_2<\cdots<n_k)$ are negative. Let $b_j=a_j$ if $a_j>0$, $b_j=0$ if $a_j<0$, and let $c_j=0$ if $a_j>0$, $c_j=-a_j$ if $a_j<0$. Then, according to J. J. Bret [Ann. Math. Pures Appl. 6, 112–122 (1815)], no positive zero of $f(x)$ exceeds $1+P$, where P is either of the limits

$$\max_{1\leq r\leq k} \frac{c_n}{b_0+b_1+\cdots+b_{n_r}}, \quad \max_{1\leq r\leq k}\left(\frac{c_{n_r}}{a_0}\right)^{1/(n_r-r+1)}.$$

In his note, using the linear combinations of positive coefficients

$$S(j,k)=\sum_{i=0}^{k-j}\binom{k-1-i}{j-1}b_i,$$

$$T(j,k,l)=\binom{l-k}{j}S(1,k)+\sum_{i=1}^{j}\binom{l-k-i}{j-i}b_{k+i},$$

Sergescu asserts that Bret's methods also lead to the more general limits for P:

$$\max_{1\leq r\leq k}\left(\frac{c_{n_r}}{S(j,n_r)}\right)^{1/(j-1)},$$

$$\max_{1\leq r\leq k}\ \min_{n_{r-1}\leq j\leq n_r}\left(\frac{c_{n_r}}{T(j-n_{r-1},n_{r-1},n_r)}\right)^{1/(n_r-j)},$$

$$\max_{1\leq r\leq k}\left(\frac{c_{n_r}}{T(\nu_r-1,\sigma_{r-1},n_r)}\right)^{1/(n_r-\sigma_r+1)}.$$

In the above formulas, $n_0=0$, $S(1,0)=0$ and $\nu_1, \nu_2, \cdots, \nu_k$ are any positive integers such that $\sigma_r=\nu_1+\nu_2+\cdots+\nu_r \leqq n_r$. In his note, Montel suggests that the above limits can be derived by use of the following principle: If $P_r(x)$, a real polynomial with highest coefficient positive, has l_r as the upper limit of its positive zeros and if $P_r(x)>0$, all r, implies $f(x)>0$, then $f(x)\neq 0$ for $x>\max l_r$. For example, Sergescu's first limit may be derived by taking $r=1, 2, \cdots, k$ and

$$P_r(x)=(x-1)\sum_{i=0}^{n_r-1}b_i x^{n_r-1-i}-c_{n_r}.$$

M. Marden (Madison, Wis.).

Markoff, A. On the determination of the number of roots of an algebraic equation, situated in a given domain. Rec. Math. [Mat. Sbornik] N.S. **7 (49)**, 3–6 (1940). (English. Russian summary) [MF 2276]

Problem: Let $f, f_1, \cdots, f_m$ be $m+1$ polynomials with rational coefficients. To find the number of real roots ξ of $f(\xi)=0$ satisfying $f_j(\xi)>0$ $(j=1, \cdots, m)$. (I) For $m=1$, solved by Hermite and Sylvester. (II) Reduction to the case f relatively prime to each f_j. (III) Let A be an arbitrary subset of set $(1, \cdots, m)$, p_A the number of real roots of $f=0$ satisfying $\Pi_{j\in A} f_j(\xi)>0$, taking vacuous sets A into account by appropriate assumptions. The determination of p_A is of the type of the general problem for $m=1$. (IV) The final step consists in expressing the number of roots of $f=0$ satisfying $f_1>0, \cdots, f_m>0$ as a function of the numbers p_A.

It is not clear to the reviewer at which step the restriction on the coefficients to be rational, rather than real, is required. *A. J. Kempner* (Boulder, Colo.).

Kharadse, A. Eine Anwendung des Graceschen Faltungssatzes. Mitt. Georg. Abt. Akad. Wiss. USSR [Soobščenia Gruzinskogo Filiala Akad. Nauk SSSR] **1**, 175–180 (1940). (Russian. German summary) [MF 2022]

Let α be a primitive κth root of unity and $f(z)$ a polynomial of degree m. The author proves that, if $f(1)=f(\alpha)=\cdots=f(\alpha^{\kappa-1})$ and $f'(1)+f'(\alpha)+\cdots+f'(\alpha^{\kappa-1})=0$, then there is a root of the third derivative in every circle (the boundary included) containing all the roots of the equation

$$\sum_{l=0}^{\kappa-1}(\alpha^l-z)^{m-1}\left[\sum_{l=0}^{\kappa-1}\alpha^{\kappa-l}(\alpha^l-z)^m\right]^{-1}=1/m.$$

In particular for $m=2\kappa+1$ or $2\kappa+2$ there is a root of the third derivative in the circle about the origin of radius

$$[(\kappa-1)(\kappa+1)]^{\frac{1}{2}}[(^{2\kappa}_{\kappa}{}^{-2})(2\kappa-1)(2\kappa+1)]^{-\frac{1}{2}}$$

or

$$[(\kappa-1)(\kappa+1)]^{\frac{1}{2}}[(^{2\kappa-1}_{\kappa+1})(2\kappa+1)]^{-\frac{1}{2}},$$

respectively. This theorem is similar to that of Grace-Heawood [P. J. Heawood, Quart. J. Math. **38**, 84–107 (1907)]. Here, however, one is interested in the roots of the third derivative, and the polynomial is subject to conditions not necessarily at only two points but at the κ vertices of a regular polygon. *A. E. Ross* (St. Louis, Mo.).

Tschebotareff, N. G. On continuable polynomials. I. General statement of the problem. Memorial volume dedicated to D. A. Grave [Sbornik posvjaščenii pamjati D. A. Grave], Moscow, 1940, pp. 268–282. (Russian) [MF 3520]

The author states the general problem which consists in establishing conditions (necessary and sufficient) for the quantities $a_0, a_1, \cdots, a_n$ in order that there exists a polynomial of the form $a_0+a_1z+\cdots+a_nz^n+b_1z^{n+1}+\cdots+b_mz^{n+m}$ with the given property X. Gavriloff has solved the problem when X is the property of having all the roots on $|z|=1$. Some solutions for the problem on R-continuable polynomials stated by the author are given. However, the corresponding general case is solved in Meiman's paper [see the following review]. Other kinds of properties X are considered. *S. Mandelbrojt* (Houston, Tex.).

Meiman, N. N. On continuable polynomials. II. On R-continuable polynomials. Memorial volume dedicated to D. A. Grave [Sbornik posvjaščenii pamjati D. A. Grave], Moscow, 1940, pp. 117–165. (Russian) [MF 3510]

The author solves the following problem on R-continuable polynomials stated by Tschebotareff [cf. the preceding review]. Given a necessary and sufficient condition for the $n+1$ real quantities $1, a_1, a_2, \cdots, a_n$, in order that there exists a polynomial $1+a_1z+a_2z^2+\cdots+a_nz^n+b_1z^{n+1}+\cdots+b_mz^{n+m}$ with real roots. He gives the smallest m, and actually constructs the corresponding polynomial. This problem can also be stated: find conditions for $s_1, s_2, \cdots, s_n$ such that there exist $m+n$ $(m\geqq 0)$ numbers $t_1, t_2, \cdots, t_{m+n}$ with the properties $s_k=\sum t_i^k$. This particular moment problem is solved by introducing "quasi-discriminant surfaces" which generalizes the notion of discriminant surfaces. *S. Mandelbrojt.*

Vassiliou, Ph. Über die Galoissche Gruppe einer Klasse von trinomischen Gleichungen. Math. Ann. **117**, 448–452 (1940). [MF 3005]

The author proves that if p is a prime then the trinomials with integral rational coefficients

$$F(x)=x^p+ax^2+b$$

have the symmetric group for their Galois group provided $F(x)$ is irreducible and certain divisibility conditions of the coefficients are satisfied. The method of proof depends upon certain relations between prime ideal properties and the substitutions of the group and certain theorems on Newton polygons. *O. Ore* (New Haven, Conn.).

Jakovkin, M. V. Sur un critérium d'irréductibilité de polynômes. C. R. (Doklady) Acad. Sci. URSS (N.S.) **28**, 771–773 (1940). [MF 3615]

Let $f(x)$ be a polynomial of degree n with integral coefficients; let x_1 and x_2 be rational numbers, where $x_1>0$ and x_2 is greater than the real parts of the roots of $f(x)$; finally let M be the lowest common multiple of the denominators of x_1 and x_2. The author proves that $f(x)$ is irreducible in the field of rational numbers if $M^n f(x_1+x_2)$ is a prime. This theorem, whose proof is very simple, is a generalization of a similar criterion due to Pólya [Pólya und Szegö, Aufgaben und Lehrsätze, vol. II, 1929, p. 137, problem no. 127]. *H. W. Brinkmann* (Swarthmore, Pa.).

Nikol, Friedrich. Über die Extremwerte von Polynomen. Jber. Deutsch. Math. Verein. **50**, Abt. 2., 15–17 (1940). [MF 2777]

Let $y=\sum A_\nu x^{n-\nu}$ be a polynomial of degree n with real coefficients and with $n-1$ extremal points. The author finds for even polynomials of degree 2, 4, 6 and 8 various relations between the coordinates of the extremal points and the coefficients A_ν. *A. C. Schaeffer.*

Macintyre, A. J. and Fuchs, W. H. J. Inequalities for the logarithmic derivatives of a polynomial. J. London Math. Soc. **15**, 162–168 (1940). [MF 3133]

The authors obtain upper bounds for the absolute values of sums $\sum_{k=1}^{n}(z-a_k)^{-m}$, where the a_k are arbitrary complex numbers and m is a positive integer. By applying H. Cartan's lemma on polynomials [Ann. École Norm. (3) **45**, 255–346 (1928); on pp. 272–276 the authors give a proof of the variant form which they need], they show that, if $H>0$ and $\alpha>0$ are arbitrary, there exists a set of at most n circles, whose radii h_k satisfy $\sum h_k^{1/\alpha}\leqq(2H)^{1/\alpha}$, such that, if z is outside these circles,

$$(1)\qquad \left|\sum_{k=1}^{n}\frac{1}{(a_k)^m-z}\right|<\frac{n^{m\alpha}}{H^m}\sum_{p=1}^{n}\frac{1}{p^{m\alpha}}.$$

The authors are interested only in the cases $\alpha=1$, $\alpha=\frac{1}{2}$; if $m\alpha\neq 1$, (1) is best possible (with respect to order in n

and H); but if $m\alpha=1$, the right side of (1) is of order $n \log n$ in n. The authors conjecture that $n \log n$ can be replaced by n. In support of their conjecture, they show that, when z is restricted to a straight line,

$$\left|\sum_{k=1}^{n} \frac{1}{z-a_k}\right| < H$$

except perhaps for z in a set (on the line) of measure less than $2en/H$. The proof depends on the theory of subharmonic functions. The authors further conjecture that the factor e can be dropped. *R. P. Boas, Jr.*

Linear Algebra

Ivanov, V. On the convergence of the process of iteration in the solution of a system of linear algebraic equations. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **1939**, 477–483 (1939). (Russian. English summary) [MF 2091]

Writing the system of equations in the matrix form $x=u+Ax$, the author shows very simply that the process of simple iteration ($x^{(\nu)}=uA^{(\nu-1)}$) converges if and only if the absolute values of the characteristic roots of A are less than unity. He notes that Seidel's [Abh. Bayer. Akad. Wiss. **11**, Abt. 3, p. 81 (1874)] process of iteration for the original system is equivalent to simple iteration in a related system and points out the existence of systems in which one of the two processes converges while the other one diverges. *A. E. Ross* (St. Louis, Mo.).

Gomes, Ruy Luís. Some applications of the notion of a matrix associated with a vector. Portugaliae Math. **1**, 293–302 (1940). (Portuguese) [MF 2225]

The matrix associated with vector $x(x_1, x_2, x_3)$ is

$$X(x)=\begin{vmatrix} x_3 & x_1-ix_2 \\ x_1+ix_2 & -x_3 \end{vmatrix}.$$

With three vectors x, y, z the system of the following equations is formed:

$$X(x)X(y)+X(y)X(z)=2(x, y) \text{ cycl.},$$
$$X(x)X(y)-X(y)X(z)=2iX(x \wedge y) \text{ cycl.},$$

where (x, y) is the scalar product and $x \wedge y$ the vector product. The author investigates certain properties of these systems and the behavior of $X(x)$ under a rotational transformation of x. *D. J. Struik* (Cambridge, Mass.).

Landherr, W. Algebraischer Beweis eines Satzes aus der Matrizenrechnung. Monatsh. Math. Phys. **49**, 197–198 (1940). [MF 2995]

An algebraic proof of a theorem established by Radon in Monatsh. Math. Phys. **48**, 198–204 (1939) [these Rev. **1**, 98]. *N. H. McCoy* (Northampton, Mass.).

Burington, Richard Stevens. On circavariant matrices and circa-equivalent networks. Trans. Amer. Math. Soc. **48**, 377–390 (1940). [MF 3162]

A minor matrix $A_{r_1 \cdots r_t}^{s_1 \cdots s_t}$ obtained from the $n \times n$ matrix A by deleting rows $r_1, \cdots, r_t$ and columns $s_1, \cdots, s_t$ is called circavariant if

$$B_{r_1 \cdots r_t}^{s_1 \cdots s_t}=P'_{r_1 \cdots r_t} A_{r_1 \cdots r_t}^{s_1 \cdots s_t} P_{s_1 \cdots s_t}^{s_1 \cdots s_t}$$

for every matrix $B=P'AP$, P non-singular. The case $t=1$ was previously considered by the author [J. Math. Phys. Mass. Inst. Tech. **16**, 83–103 (1937)], and applied to the problems of the equivalence of quadrics in affine m-space, and of the equivalence of electrical networks. In the present paper the author obtains conditions which must be imposed on P in order that $A_{r_1 \cdots r_t}^{s_1 \cdots s_t}$ be circavariant. The case where P is m-affine is of particular importance, and normal forms characterized by invariants are determined for this case. The Kirchhoff equations for an m-terminal pair bilateral n-mesh linear electrical network may be written $\{Q\}=C\{E\}$. The matrix Y, obtained from C by deleting all but the first m rows and columns, is a characteristic (admittance) coefficient matrix for the network. Two networks are circa-equivalent if their corresponding matrices are related by $Y^{(1)}=D'Y^{(2)}D$, where D is real, diagonal and non-singular. *C. C. MacDuffee* (New York, N. Y.).

Browne, E. T. On the reduction of a matrix to a canonical form. Amer. Math. Monthly **47**, 437–450 (1940). [MF 2826]

An exposition, using the notion of chains of vectors, of the reduction of a matrix with elements in a field F (1) to the rational canonical form, (2) to the classical canonical form (in case F is algebraically closed) and (3) to the Jacobson rational canonical form [Notes on modern higher algebra (mimeographed), University of North Carolina]. *N. H. McCoy* (Northampton, Mass.).

Taussky, Olga and Todd, John. Matrices with finite period. Proc. Edinburgh Math. Soc. (2) **6**, 128–134 (1940). [MF 2843]

A matrix A of order n with rational integral elements is of period r if A^r is the identity matrix and r is minimal. For a given n only a finite number of values of r exist, and if A is irreducible in the rational field, $\varphi(r)=n$. If A is irreducible, the only possible values of r occur among the orders of the roots of unity generating algebraic fields of degree n, and all such r's can be realized. If r is the order of a unit in the ring of integers of a group of order n, then $r \leqq$ a constant depending only on n. *C. C. MacDuffee.*

Specht, Wilhelm. Zur Theorie der Matrizen. II. Jber. Deutsch. Math. Verein. **50**, 19–23 (1940). [MF 2770]

With any function $G(x, y)=x^{\alpha_1}y^{\beta_1}x^{\alpha_2}y^{\beta_2} \cdots x^{\alpha_n}y^{\beta_n}$, where the exponents α_i, β_i are non-negative integers, may be associated a matrix function $G(X, Y)$ of the matrices X, Y of given order m; denote the trace of this matrix by $\sigma G(X, Y)$. The author proves that a necessary and sufficient condition for the unitary equivalence of two matrices A, B, that is, for $U^*AU=B$ where $U^*=\bar{U}'$, is that $\sigma G(A, A^*)=\sigma G(B, B^*)$ for every function G. He also considers the reduction of a matrix in the sense of unitary equivalence [cf. Jber. Deutsch. Math. Verein. **49**, 207–215 (1940); these Rev. **1**, 162]. *G. de B. Robinson* (Toronto, Ont.).

Specht, Wilhelm. Klassifikation der halblinearen Transformationen. Math. Z. **46**, 637–649 (1940). [MF 3373]

Let $\mathbf{Z}$ be a cyclic field of degree k over a field $\mathbf{K}$ of characteristic zero. From any matrix $X=(\xi_{\mu\nu})$ which is rational in $\mathbf{Z}$, we may obtain a matrix $X^G=(\xi_{\mu\nu}^G)$ by applying an automorphism G of $\mathbf{Z}$ to the elements of X. Two matrices A, B which are rational in $\mathbf{Z}$ are said to be G-similar if $P^{-1}AP^G=B$. Nakayama proved [Proc. Phys.-Math. Soc. Japan **19**, 99–107 (1937)] that a class (A, G) of G-similar matrices is characterized by the invariant factors (elementarteiler) of the matrix $N(A)=AA^GA^{G^2} \cdots A^{G^{k-1}}$ and the

ranks of the matrices A, AA^G, $AA^GA^{G^2}$, $\cdots$, $AA^G\cdots A^{G^{k-1}}$. Specht gives a proof of this theorem based on purely matrix theory considerations. *G. de B. Robinson.*

Fischer, Otto F. Lorentz transformation and Hamilton's quaternions. Philos. Mag. (7) **30**, 135–150 (1940). [MF 2767]

The author obtains the relation between Lorentz transformations and the mappings of quaternion algebra on itself. This relation is known, for it has been recognized that the algebra of two-component spinors with one dotted and one undotted index is simply related to the algebra of quaternions over the complex field. The author also obtains relations equivalent to the well-known relations between self-dual anti-symmetric four-dimensional tensors and symmetric two-component second-order spinors.

A. H. Taub (Princeton, N. J.).

Oldenburger, Rufus and Porges, Arthur. The minimal numbers of binary forms. Bull. Amer. Math. Soc. **46**, 694–697 (1940). [MF 2661]

If an algebraic form F of degree m over a field K is written as a linear combination of σ mth powers of linear forms, where σ is minimal, then σ is called the minimal number of F with respect to K. For a field K with characteristic 0 or greater than m containing at least $\frac{1}{2}m(m^2-m+2)$ elements, the range of the minimal numbers of binary forms of degree m is $1, 2, \cdots, m$. *C. C. MacDuffee* (New York, N. Y.).

Oldenburger, Rufus. Binary forms. Proc. Nat. Acad. Sci. U. S. A. **26**, 497–499 (1940). [MF 2636]

Consider a binary form $G=L_1\cdots L_\sigma$, where $L_1, \cdots, L_\sigma$ are distinct linear forms with coefficients in a field K. It is known that a binary form F of degree n over K can be given a "σ-representation"

$$\lambda_1 L_1^n+\lambda_2 L_2^n+\cdots+\lambda_\sigma L_\sigma^n, \qquad \sigma\leqq n,$$

where the λ_i are in K, if and only if G and F are apolar. In the present paper the author determines the values of σ for which F can be written as above, and determines for each σ the class of forms G for which F can be so written. In particular, he arranges the elements of the symmetric tensor A of coefficients of F in an ordinary matrix display A_σ whose column index is the set of σ-indices of A. He introduces a covariant F_σ whose coefficients are, except for sign, identical with the components of the vector ξ, where $A\xi=0$. The form F has a σ-representation as defined above if and only if F_σ can be written for some choice of its coefficients as the product $L_1\cdots L_\sigma$. If $F_\sigma\neq 0$ and $\rho\geqq\sigma$, the form F_σ divides F_ρ. The minimal number of F is given in terms of the ranks of $A_0, \cdots, A_n$ and the factorization properties of $F_0, \cdots, F_n$. The ranks of $A_0, \cdots, A_n$ depend only on the value of γ, which is the smallest integer such that $F_\gamma\neq 0$. A σ-representation is minimal if $\sigma\leqq(n/2)+1$, and unique if $\sigma\leqq(n+1)/2$. *C. C. MacDuffee* (New York, N. Y.).

Oldenburger, Rufus. The theory of polynomials of higher order. Ingenieria (Mexico City) **14**, 192–201, 239–244 (1940). (Spanish) [MF 2743]

This paper consists of five lectures delivered at the National University of Mexico. The first lecture is elementary. The second contains a proof of the existence of representations, the third and fourth are elaborations with proofs and examples of the two papers reviewed above. The fifth contains applications to the problems of the factorization and equivalence of polynomials. *C. C. MacDuffee.*

Oldenburger, Rufus. Polynomials over fields. Nat. Math. Mag. **15**, 1–24 (1940). [MF 3441]

This is an elementary exposition of the author's previously published results on the representation of an algebraic form as a sum of powers of linear forms.

C. C. MacDuffee (New York, N. Y.).

Habicht, W. Über die Zerlegung strikte definiter Formen in Quadrate. Comment. Math. Helv. **12**, 317–322 (1940). [MF 3060]

Let us denote by "form" a homogeneous polynomial with rational coefficients; the variables are supposed to run through all real values. A form that never assumes a negative value is called definite; a form that always assumes positive values, except of course at the point $(0, 0, \cdots, 0)$, is called strictly definite. Hilbert stated and proved for ternary forms, and Artin proved generally, the theorem that each definite form is the sum of the squares of certain homogeneous rational functions (not necessarily polynomials!) [see Abh. Math. Sem. Hansischen Univ. **5**, 100–115 (1926)]. The author proves the much more special theorem that results from the foregoing when "definite form" is replaced by "strictly definite form." His proof is essentially different from Artin's proof, more elementary and constructive, and reaches the conclusion in a few surprisingly simple steps, starting from a foregoing theorem of Pólya [Vierteljschr. Naturforsch. Ges. Zürich **73**, 141–145 (1928) or Hardy, Littlewood and Pólya, Inequalities, Cambridge, 1934, pp. 57–60]. *G. Pólya* (Providence, R. I.).

Mahler, K. On reduced positive definite ternary quadratic forms. J. London Math. Soc. **15**, 193–195 (1940). [MF 3137]

Let

$$f(x)=a_1x_1^2+a_2x_2^2+a_3x_3^2+2b_1x_2x_3+2b_2x_1x_3+2b_3x_1x_2$$

be a positive definite ternary quadratic form with real coefficients. It is called reduced if its coefficients satisfy the inequalities $0<a_1\leqq a_2\leqq a_3$, $0\leqq b_1\leqq\frac{1}{2}a_2$, $|b_2|\leqq\frac{1}{2}a_1$, $0\leqq b_3\leqq\frac{1}{2}a_1$, $b_1-b_2+b_3\leqq\frac{1}{2}(a_1+a_2)$. The author gives a new and very simple proof for the well-known theorem that, for reduced forms, $a_1a_2a_3\leqq 2D$, where

$$D=a_1a_2a_3-(a_1b_1^2+a_2b_2^2+a_3b_3^2-2b_1b_2b_3)$$

is the determinant of $f(x)$. *P. Erdös* (Philadelphia, Pa.).

Hestenes, M. R. and McShane, E. J. A theorem on quadratic forms and its application in the calculus of variations. Trans. Amer. Math. Soc. **47**, 501–512 (1940). [MF 2164]

The general result of this paper is as follows: suppose P, Q_i $(i=1, \cdots, r)$ are real quadratic forms in $z_1, \cdots, z_m$ such that: (i) $P>0$ at each point $(z)\neq(0)$ satisfying $Q_i=0$ $(i=1, \cdots, r)$; (ii) for arbitrary constants $S_1, \cdots, S_r$ not all zero the form $\sum_i S_iQ_i$ is indefinite; (iii) for arbitrary linear subspaces L on which $Q_1, \cdots, Q_r$ do not vanish simultaneously, save at $(z)=(0)$, there exists a linear combination $\sum_i S_iQ_i$ which is positive definite on L. Then there exist constants $S_1, \cdots, S_r$ such that $P+\sum_i S_iQ_i$ is positive definite; if $r=1$ hypotheses (ii) and (iii) may be omitted. For the case $r=1$ proofs of this theorem have been published by Albert [Bull. Amer. Math. Soc. **44**, 250–252 (1938)] and Reid [ibid., 437–440]. This general theorem is used to prove the following result: if P is a real quadratic form in $2n$ variables $(x_1, \cdots, x_n, y_1, \cdots, y_n)$ and $P>0$ whenever the $n\times 2$ dimensional matrix (x_i, y_i) has rank 1, then there

exists an n-rowed skew-symmetric matrix S such that the form $P+\sum_{i,k=1}^{n} S_{ik}x_i\sigma_k$ is positive definite. Finally, the significance of this latter result for the Legendre condition in general double integral problems of the calculus of variations is discussed. *W. T. Reid* (Chicago, Ill.).

Deruyts, J. Sur la théorie des formes algébriques. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **25**, 360–374 (1939). [MF 2542]

Let (y), (z), $\cdots$ be n sets of variables cogredient to the set $x=(x_1, x_2, \cdots, x_n)$ and let $f, f_1, \cdots$ be forms in these variables. Further let p be a rational integral function of the variables (y), (z), $\cdots$, and of the coefficients of the forms $f, f_1, \cdots$. The author shows that, if the transforms of p are the same for the products ST and TS of two arbitrary linear substitutions, then p is the sum of invariants of the forms $f, f_1, \cdots$, together with a constant. In proving this the author replaces the matrix of the general substitution T by the sum of the identity matrix and the matrix whose only non-zero element is an arbitrary parameter ϵ in the ith row and kth column. Then he shows that, if p is homogeneous, p is an invariant. From this, by the usual argument, he deduces the general result. Then, by similar methods, he proves that, if the transform of p by ST is the same as that by TS, when certain rational relations exist among the coefficients of the forms f, then p is the sum of invariants and a function which reduces to a constant, when the relations that exist among the coefficients are satisfied. He also shows that this last result is true if the transformations ST and TS are replaced by S and the transposed of S, respectively. Finally he determines conditions, involving homogeneity, isobaricness and a single substitution, that p be an invariant. *J. Williamson* (Baltimore, Md.).

Abstract Algebra

***van der Waerden, Bartel Leendert. Moderne Algebra.** Bd. 2. 2nd ed. J. Springer, Berlin, 1940. viii+224 pp. 16.50 RM.

Church, Randolph. Numerical analysis of certain free distributive structures. Duke Math. J. **6**, 732–734 (1940). [MF 2726]

The author summarizes here his investigations of the free distributive lattice Δ_n on n elements, $n\leqq 5$. Two elements of Δ_n are called "conjugate" if there exists a lattice automorphism throwing one into the other. Conjugacy is an equivalence relation preserving rank. Hence Δ_n is determined if we know the rank, number of elements and typical element of each of its conjugate classes. The author tabulates this information and the order $N(\Delta_n)$ of Δ_n for $n\leqq 5$. All the results for $n=5$ are new. The value 7579 for $N(\Delta_5)$ disproves a conjecture that $N(\Delta_n)+2$ is always divisible by $(2n-1)(2n-2)$ [G. Birkhoff, Lattice Theory, New York, 1940, p. 85]. *M. Ward* (Pasadena, Calif.).

Dilworth, R. P. On complemented lattices. Tôhoku Math. J. **47**, 18–23 (1940). [MF 2619]

After defining relative complement in a manner that is not self-dual, it is proved that, if every relatively complemented sublattice of a relatively complemented lattice S of finite dimensions satisfies the Jordan-Dedekind chain law, then S is the dual of a Birkhoff lattice. It follows that, if S has a negation (is orthocomplemented), then S is modular. The conjecture of Husimi that a lattice with negation is modular if the chain law holds for every sublattice closed with respect to relative negation is shown by an example to be false. It is also proved that every complemented nonmodular lattice L of finite dimensions contains a complemented nonmodular sublattice of five elements, with the same unit and zero elements as L. This yields a new proof of the theorem of G. Birkhoff and M. Ward that a lattice of finite dimensions is a Boolean algebra if and only if every element has a unique complement. *O. Frink* (State College, Pa.).

Dilworth, R. P. Lattices with unique irreducible decompositions. Ann. of Math. (2) **41**, 771–777 (1940). [MF 3020]

Let L be a lattice with ascending chain condition. Ideal theory motivates defining a reduced decomposition into irreducibles of an element a, as an expression $a=x_1\cap\cdots\cap x_n$, where (1) each x_i is irreducible, in the sense that $x_i=y\cap z$ implies $y=x_i$ or $z=x_i$, and (2) a is not the meet of any proper subset of the x_i. The existence of such decompositions is implied trivially by the ascending chain condition. The author's main result is a necessary and sufficient condition that every a in L have a unique such decomposition; this occurs if and only if (1) L is upper semi-modular (a "Birkhoff lattice"), that is, if b and c cover a, then $b\cup c$ covers b and c, and (2) every modular sublattice of L is distributive. The author points out various corollaries of this result, including the known theorems that L has unique meet- and join-decompositions if and only if it is distributive, and that, if modular, then L has unique meet-decompositions if and only if it is distributive. *G. Birkhoff*.

Ward, Morgan. Residuated distributive lattices. Duke Math. J. **6**, 641–651 (1940). [MF 2720]

The paper continues researches of the author and R. P. Dilworth on abstract ideal theory [cf. Ward-Dilworth, Trans. Amer. Math. Soc. **45**, 335–354 (1939)]. Under addition (or union) and intersection, ideals form a lattice; the author studies the case that this lattice is distributive, satisfies the ascending chain condition, and satisfies certain axioms always satisfied by ideals relative to multiplication, addition and intersection. If $a\geqq b$ implies $ax=b$ for some x (implying distributivity), every a can be written uniquely as the intersection and also as the product of prime-powers. There exist distributive "Noether lattices" (in which every "irreducible" element is "primary"), in which decomposition of elements into the intersection of "primary" components is not unique; for this, the condition $(b\cup c):a=b:a\cup c:a$ is however sufficient and, moreover, equivalent to the assertion that any two primaries are either coprime, or else one divides the other. Other results of the same nature are proved. *G. Birkhoff* (Cambridge, Mass.).

Birkhoff, Garrett. Neutral elements in general lattices. Bull. Amer. Math. Soc. **46**, 702–705 (1940). [MF 2663]

The notion of neutral element, introduced by Ore for modular lattices, is extended to general lattices. An element a is called neutral if every triple $\{a, x, y\}$ generates a distributive sublattice. Neutral elements of a lattice L are characterized as those elements which correspond to the pair $[I, O]$ in some representation of L as a sublattice of a direct product of two lattices. It is shown that the set of all neutral elements of L is a distributive sublattice of L,

and is, in fact, the intersection of all maximal distributive sublattices of L.

Complements of neutral elements, when they exist, are shown to be unique and neutral. The complemented neutral elements of L form a Boolean algebra called the center of L. The center is shown to consist of those elements which correspond to the pair $[I, O]$ in some representation of L as a direct product. The first formula in the proof of Lemma 1 is misprinted. It should read $(x \cup y) \cap a = (x \cap a) \cup (y \cap a)$.

O. Frink (State College, Pa.).

Mori, Shinziro. Allgemeine Z.P.I.-Ringe. J. Sci. Hirosima Univ. Ser. A. **10**, 117–136 (1940). [MF 2947]

A commutative ring $\mathfrak{R}$ is termed a general Z.P.I. ring if every ideal in $\mathfrak{R}$ can be expressed as a product of a finite number of prime ideals. Thus rings without unit element for multiplication and rings with divisors of zero are included in the class of rings considered by the author. As a main result the author proves that a ring $\mathfrak{R}$ is a Z.P.I. ring if and only if (1) every ideal of $\mathfrak{R}$ has a finite basis, (2) for every pair of maximal prime ideals $\mathfrak{P}$, $\mathfrak{P}'$ (that is, $\mathfrak{R}|\mathfrak{P}$, $\mathfrak{R}|\mathfrak{P}'$ are fields $\neq 0$) there is no ideal $\mathfrak{Q}$ with $\mathfrak{P}\mathfrak{P}' \subset \mathfrak{Q} \subset \mathfrak{P}$, (3) there is no ideal $\mathfrak{Q}$ with $\mathfrak{R}^2 \subset \mathfrak{Q} \subset \mathfrak{R}$. (The three conditions are independent.) This theorem essentially depends on the fact that in a Z.P.I. ring $\mathfrak{P}\mathfrak{P}_1 = \mathfrak{P}$ if $\mathfrak{P} \subset \mathfrak{P}_1$ and if $\mathfrak{R}|\mathfrak{P}$ is not a field. To prove the latter assertion it is necessary to investigate the relationship between the ideal theory of $\mathfrak{R}$ and $\mathfrak{R}|\mathfrak{P}$. Finally the author formulates two theorems which are equivalent to his main theorem. For details and the methods of proof see the original paper.

O. F. G. Schilling (Chicago, Ill.).

Schneidmüller, V. I. On rings with finite decreasing chains of subrings. C. R. (Doklady) Acad. Sci. URSS (N.S.) **28**, 579–581 (1940). [MF 3610]

A ring R is said to satisfy the chain condition of decreasing sequences if every chain of subrings $R \supset R_1 \supset \cdots \supset R_i \supset R_{i+1} \cdots$ breaks off at a finite place. The author proves that the additive group of such a ring is periodical. Moreover, the radical $\mathfrak{R}$ is at most denumerable. Defining a ring to be locally finite if every finite subset generates a finite subring it is shown that a ring R satisfying the chain condition is locally finite if and only if the difference ring $R|\mathfrak{R}$ does not contain non-commutative fields. The author's results permit a rephrasing of Burnside's problem on the periodicity of locally finite groups. *O. F. G. Schilling.*

Zassenhaus, Hans. Darstellungstheorie nilpotenter Lie-Ringe bei Charakteristik $p>0$. J. Reine Angew. Math. **182**, 150–155 (1940). [MF 3428]

After summarizing some of the results of a former paper [Abh. Math. Sem. Hansischen Univ. **13**, 1–100 (1939)], the author gives a method for constructing a representation of a nilpotent Lie-ring over a field of characteristic $p>0$, and deduces the fact that the degree of each irreducible representation is a power of p. *G. de B. Robinson.*

Gertschikoff, A. I. Über Ringe, die in eine direkte Summe von Körpern zerlegbar sind. Rec. Math. [Mat. Sbornik] N.S. **7** (**49**), 591–597 (1940). (Russian. German summary) [MF 2807]

v. Neumann has called a ring regular [Proc. Nat. Acad. Sci. U. S. A. **22**, 707–713 (1936), incorrectly quoted as Proc. Amer. Math. Soc.] if for every a there is an x such that $a = axa$. Making strong use of this concept, it is shown that a ring K is the two-sided direct sum of fields if and only if (1) the minimal-condition for left principal ideals is satisfied in K, and (2) K contains no nilpotent elements.

H. Wallman (Chapel Hill, N. C.).

Neumann, B. H. On the commutativity of addition. J. London Math. Soc. **15**, 203–208 (1940). [MF 3139]

Because of applications to the theory of affine and projective geometries, it is important to know that the commutative law of addition follows from the other rules in certain algebraic systems having only a one sided distributive law. Zassenhaus proved results of this sort for finite algebras. The author proves that addition is commutative in any algebra S (not necessarily finite) which has the following properties: S is a group with respect to addition, multiplication is associative and distributive on the right, equal non-zero left-hand factors may be cancelled, and there exists a non-zero idempotent element which can be halved. He shows this by proving the more general result that a group G is necessarily abelian if it has an automorphism of period two, which leaves no element except the identity unchanged, and the equation $x^2 = a$ has one and only one solution x for every a in G. It is shown by examples that none of these conditions on G can be omitted.

O. Frink (State College, Pa.).

Abe, Makoto. Eine Bemerkung über einfache Systeme. Proc. Imp. Acad. Tokyo **16**, 384–386 (1940). [MF 3481]

Let k be a perfect field. A finite k-module $\mathfrak{s}$ is termed a distributive "k-system" if for every pair of elements x, y in $\mathfrak{s}$ there is defined a distributive product xy such that

$$(\alpha_1 x_1 + \alpha_2 x_2)y = \alpha_1(x_1 y) + \alpha_2(x_2 y),$$
$$x(\alpha_1 y_1 + \alpha_2 y_2) = \alpha_1(x y_1) + \alpha_2(x y_2)$$

with α_1, α_2 in k. A k-system $\mathfrak{s}$ is called simple if $\mathfrak{s}$ possesses no proper two-sided subideals. Moreover, a simple k-system $\mathfrak{s}$ is defined to be normal simple if the direct product of $\mathfrak{s}$ with the algebraic closure of k is simple. The author proves a generalization of a theorem of W. Landherr [Abh. Math. Sem. Hansischen Univ. **11**, 44–64 (1936)]: "A simple k-system $\mathfrak{s}$ over a perfect field k is k-operator isomorphic to a normal simple system $\mathfrak{t}$ over a finite extension Ω of k. The field Ω and the Ω-system $\mathfrak{t}$ are uniquely determined to within isomorphisms." For related results cf. also Jacobson [Duke Math. J. **3**, 544–548 (1937)]. *O. F. G. Schilling.*

Witt, Ernst. Die Automorphismengruppen der Cayleyzahlen. J. Reine Angew. Math. **182**, 205 (1940). [MF 3434]

Bericht über eine Dissertation von Fräulein E. Bannow [Abh. Math. Sem. Hansischen Univ. **13**, 240–256 (1940); cf. these Rev. **1**, 328]. *Author's summary.*

Etherington, I. M. H. Commutative train algebras of ranks 2 and 3. J. London Math. Soc. **15**, 136–149 (1940). [MF 2523]

The author continues earlier work [Proc. Roy. Soc. Edinburgh **59**, 242–258 (1939); these Rev. **1**, 99] on non-associative linear algebras of types occurring in genetics, namely, baric algebras (those in which each element may be assigned a weight in the base field so that the correspondence from element to weight is a non-trivial homomorphism) and train algebras (those baric algebras in which the rank equation satisfied by the general element has coefficients depending only on the weight of that element). In this paper, the multiplication tables for all commutative baric algebras are determined; and the nature of all commutative train alge-

bras of rank 2 and of all commutative train algebras of rank 3 with a certain exception is fully investigated by reduction of their multiplication tables to simple canonical forms. The discussion involves the properties of the nil-product (defined by the equation $x\cdot y=xy-\frac{1}{2}\xi y-\frac{1}{2}\eta y$, where x and y have ξ and η as their respective weights) and of the similarly defined triple nil-product. *M. H. Stone.*

Hall, Marshall. The position of the radical in an algebra. Trans. Amer. Math. Soc. **48**, 391–404 (1940). [MF 3163]

Let $\mathfrak{A}$ be a linear associative algebra over a field K, and let $\mathfrak{R}$ be its radical. The elements c of $\mathfrak{A}$ which are left and right annihilators of $\mathfrak{R}$ form a two-sided ideal $\mathfrak{R}'$ of $\mathfrak{A}$. An algebra is said to be a bound algebra if $\mathfrak{R}'\subseteqq\mathfrak{R}$. It is shown that an arbitrary algebra can be represented as a direct sum $\mathfrak{A}=\mathfrak{B}\oplus\mathfrak{C}$, where $\mathfrak{B}$ is semisimple and $\mathfrak{C}$ is a bound algebra; both $\mathfrak{B}$ and $\mathfrak{C}$ are uniquely determined. Because of this result, the further investigation can be confined to bound algebras. The following properties of bound algebras may be mentioned. In a bound algebra $\mathfrak{A}$, there are three subalgebras $\mathfrak{A}_l$, $\mathfrak{A}_r$, $\mathfrak{A}_d$ such that $\mathfrak{A}=\mathfrak{A}_l+\mathfrak{A}_r+(\mathfrak{A}_d\cup\mathfrak{R})$ with the following properties: (1) $\mathfrak{A}_l$, $\mathfrak{A}_r$, $\mathfrak{A}_d$ have units and are orthogonal. (2) $\mathfrak{A}_l$ and $\mathfrak{A}_r$ are semisimple and $\mathfrak{R}\mathfrak{A}_l=\mathfrak{A}_l\mathfrak{R}=0$. (3) $\mathfrak{A}/\mathfrak{R}$ is the direct sum of three semisimple algebras isomorphic to $\mathfrak{A}_l$, $\mathfrak{A}_r$, $\mathfrak{A}_d/(\mathfrak{A}_d\cap\mathfrak{R})$. If the radical $\mathfrak{R}$ of a bound algebra $\mathfrak{A}$ is of order s, then the order of $\mathfrak{A}$ is at most s^2+s+1. The question arises to construct all bound algebras $\mathfrak{A}$ with a given radical $\mathfrak{R}$. This question is attacked by means of a kind of representation of $\mathfrak{A}$. Let $z_1, z_2, \cdots$ be a basis of $\mathfrak{A}$, such that the first q elements z_k form a basis of $\mathfrak{R}'$, and the first p elements z_k form a basis of $\mathfrak{R}$. For $c=\sum r_k z_k$ in $\mathfrak{A}$, we have formulae $z_i c=\sum a_{ij}z_j$, $cz_i=\sum b_{ij}z_j$, where i, j range over $1, 2, \cdots, p$. In other words, $R(c)=(a_{ij})$ and $L(c)=(b_{ij})$ are the matrices representing c in the right and left representations of $\mathfrak{A}$ belonging to the ideal $\mathfrak{R}$ of $\mathfrak{A}$. Furthermore, let $W(c)$ be the column with q rows containing the first q coefficients r_i of $c=\sum r_k z_k$. Then c is represented by the symbol $[R(c), L(c), W(c)]$ in a (1-1) manner. The properties of this type of representation are studied. The results are applied to the problem mentioned above of constructing all bound algebras with a given radical; some instructive examples are discussed. *R. Brauer.*

Carlitz, L. Linear forms and polynomials in a Galois field. Duke Math. J. **6**, 735–749 (1940). [MF 2727]

In earlier papers the author has dealt with sums and products extended over the set of primary polynomials of given degree with coefficients in a given Galois field [see for example, Duke Math. J. **5**, 941–947 (1939); these Rev. **1**, 101]. The main object of this paper is to generalize some of these things by replacing the polynomial $c_0+c_1x+\cdots+c_{m-1}x^{m-1}$ by the linear form $c_0u_0+c_1u_1+\cdots+c_{m-1}u_{m-1}$, where $u_0, u_1, \cdots, u_{m-1}$ are indeterminates. Thus there is given a formula for the sum of the $p^{nk}-1$ powers of all these forms (p^n is the order of the field) in terms of certain determinants. The methods used are similar to those employed in the earlier papers. *H. W. Brinkmann.*

Baer, Reinhold. A Galois theory of linear systems over commutative fields. Amer. J. Math. **62**, 551–588 (1940). [MF 2455]

This paper provides a different approach to results of Jacobson [Ann. of Math. (2) **41**, 1–7 (1940); these Rev. **1**, 198]. From Jacobson's theory we quote the following theorems. (a) For a quasi-field P with a finite group $\mathfrak{G}$ of outer automorphisms we have the usual (1-1) correspondence between subgroups of $\mathfrak{G}$ and subfields of P which contain the field Φ of invariant elements. (b) If moreover the automorphisms of $\mathfrak{G}$ are distinct in the center Γ of P (where they determine a field Δ of invariant elements), then P is the direct product over Δ of the field Γ (which is normal, separable, finite over Δ) and the quasi-field Φ. In other terms, P has a basis over Γ which consists of invariant elements. (c) If E is a vector space of finite rank over the quasi-field P and $\mathfrak{G}$ a finite group of semilinear transformations of E which induce distinct and outer automorphisms in P, then E has over P a basis of invariant elements.

In the present paper the quasi-field P of theorem (c) is taken commutative, and on the basis of commutative theories the following compensations are obtained. (d) No assumption about the rank of E over P is necessary. (e) The usual (1-1) correspondence prevails between the subgroups of $\mathfrak{G}$ and those linear systems between E and the system H of invariant vectors which are direct factors (that is, have a basis over their field which is also a basis of E over P). (f) The group $\mathfrak{G}$ contains all semilinear transformations of E which are idle on H. (g) The condition that the elements of $\mathfrak{G}$ induce distinct automorphisms in P is not only sufficient but also necessary for (f), at least if E has an infinite number of elements. Besides a detailed treatment of these questions and several generalizations we find also a theory of crossed products of quasi-fields with groups of automorphisms. *M. A. Zorn* (Los Angeles, Calif.).

Teichmüller, Oswald. Über die sogenannte nichtkommutative Galoissche Theorie und die Relation $\xi_{\lambda,\mu,\nu}\xi^{\lambda}_{\kappa,\mu,\nu}\xi^{\kappa}_{\lambda,\mu,\nu}=\xi_{\lambda,\mu,\nu}\xi_{\lambda\mu,\nu,\nu}$. Deutsche Math. **5**, 138–149 (1940). [MF 2815]

Let G be the group of automorphisms of an algebra A. Denote by H^* the subalgebra fixed elementwise by automorphisms of a subgroup H, and by $B^\dagger$ the subgroup leaving fixed the elements of a subalgebra B. The "noncommutative Galois theory" is concerned with finding the H's for which $(H^*)^\dagger=H$ and the B's for which $(B^\dagger)^*=B$. If A is a division algebra over a field P and if $G^*\subset P$, then $(B^\dagger)^*=B$ for all B with $P\subset B\subset A$. Equivalent to the condition that $G^*\subset P$ is the condition that every automorphism of the center Z leaving P fixed be extensible to an automorphism of A. These conditions are satisfied if certain factor sets exist. The factor sets under the customary equivalence depend solely upon the algebra class of A over P, and they may be multiplied to form a group isomorphic to the group of algebra classes over P. Application is made to prove: Let A be a simple algebra over its center Z, and let v be an automorphism of A which has finite order within Z, and let P be the field of elements of Z fixed by v. Then there is an algebra B with center P, containing A, in which only elements of A permute with Z. *M. Hall.*

Ward, James A. A theory of analytic functions in linear associative algebras. Duke Math. J. **7**, 233–248 (1940). [MF 3400]

Let $\mathfrak{A}$ be a finite linear associative algebra of order s, with unit element, over a field $\mathfrak{F}$ with n independent derivations. Let $\mathfrak{A}$ be represented by its first regular matric representation. Let $\mathfrak{M}$ be the total matric algebra of order n^t in which $\mathfrak{A}$ is embedded, and let $\mathfrak{D}$ be the algebra of lowest order ($\leqq n^2$) in which both $\mathfrak{A}$ and its reciprocal algebra $\mathfrak{A}^{-1}$ can be embedded. If $y_1, \cdots, y_n$ are functions (in $\mathfrak{F}$) of $x_1, \cdots, x_n$, then $\eta=y_1\epsilon_1+\cdots+y_n\epsilon_n$ is a function (in $\mathfrak{A}$) of $\xi=x_1\epsilon_1+\cdots+x_n\epsilon_n$, and $d\eta/d\xi$ is by definition the Jacobian matrix $(\partial y_r/\partial x_s)$. This derivative always exists in $\mathfrak{M}$

The function η is called analytic if $d\eta/d\xi$ exists in $\mathfrak{D}$. This definition is coextensive with that of Hausdorff [Ber. Verh. Sächs. Akad. Wiss. Leipzig **52**, 43–61 (1900)], who never succeeded in obtaining an accompanying definition of derivative. For two functions η and ζ,

$$\frac{d(\eta\zeta)}{d\xi}=\eta\frac{d\zeta}{d\xi}+\bar{\zeta}\frac{d\eta}{d\xi},$$

where $\bar{\zeta}$ is the correspondent of ζ under the anti-isomorphism $\mathfrak{A}\cong\mathfrak{A}$. The sum and product of analytic functions are analytic. A necessary and sufficient condition that ζ have an antiderivative in $\mathfrak{A}$ is obtained, and if $\mathfrak{A}$ is commutative, this condition reduces to the condition that ζ be analytic. A bibliography of 81 items is appended.

C. C. MacDuffee (New York, N. Y.).

MacLane, Saunders. Note on the relative structure of p-adic fields. Ann. of Math. (2) **41**, 751–753 (1940). [MF 3017]

Let k be a p-adic field whose residue class field $\mathfrak{k}$ has characteristic p. A set X of elements of $\mathfrak{k}$ is called p-independent if no element of X lies in the field obtained by adjoining all the other elements of X to $\mathfrak{k}^p$. The author proves that, if any two p-adic extensions K and K' of k, with the same residue field $\mathfrak{K}$, are analytically equivalent over $\mathfrak{K}$ and k, then every p-independent set of elements of $\mathfrak{k}$ remains p-independent in $\mathfrak{K}$. [For the converse, see MacLane, Ann. of Math. (2) **40**, 423–442 (1939), Theorem 8. Corrections not affecting Theorem 8 are required for Theorems 3 and 4 of the paper cited.] K and K' are called analytically equivalent over $\mathfrak{K}$ and k if there is an isomorphism which maps K on K', preserves the valuation, and leaves each residue class of $\mathfrak{K}$ and each element of k fixed.

R. Hull (Vancouver, B. C.).

Albert, A. A. On p-adic fields and rational division algebras. Ann. of Math. (2) **41**, 674–693 (1940). [MF 2559]

For a p-adic field $\mathfrak{K}$ of ramification order e prime to p the author obtains a canonical generation $\mathfrak{K}=W(P)$, $P^e=\zeta^i P_{\mathfrak{F}}$, $W=\mathfrak{F}(\zeta)$, where ζ is a primitive (q^f-1)st root of unity, q the number of elements in the residue field of $\mathfrak{F}$ and $ef=n=(\mathfrak{K}:\mathfrak{F})$. Two such fields are isomorphic if and only if their type numbers i are congruent mod (τ, e), $\tau=q^f-1$; $\mathfrak{K}$ is normal if and only if e is a common divisor of τ and $i(q-1)$. As a consequence of this there exists a completely ramified field of degree n prime to p over $\mathfrak{F}$ if and only if all completely ramified fields of degree n are cyclic. There is an abelian field of ramification e prime to p if and only if all such fields are abelian. The cyclic fields among these are characterized and the non-cyclic fields are shown to be direct products of pairs of cyclic fields over $\mathfrak{F}$. These results yield simple proofs of the finiteness of the number of extensions of a fixed degree of $\mathfrak{F}$ and of the solvability of every normal $\mathfrak{K}$ over $\mathfrak{F}$. Finally the author obtains an answer to the following question on division algebras: Let $\mathfrak{D}$ be a division algebra of degree n over an algebraic number field $\mathfrak{K}$ of finite degree over $\mathfrak{F}$, when does there exist a cyclic field $\mathfrak{Z}$ of degree n over $\mathfrak{F}$ whose composite with $\mathfrak{K}$ splits $\mathfrak{D}$?

N. Jacobson (Baltimore, Md.).

Krasner, Marc. La loi de Jordan-Hölder dans les hypergroupes et les suites génératrices des corps de nombres p-adiques. Chapitre II. Suites génératrices des corps de nombres p-adiques. Duke Math. J. **7**, 121–135 (1940). [MF 3393]

If $K=k(\alpha)$ is a finite algebraic extension of a field k, the set $G_{K/k}$ of isomorphisms $\sigma, \sigma_1, \sigma_2, \cdots$ of K/k with its conjugates over k is a hypergroup when the product $\sigma_1\sigma_2$ is defined to be the set of all σ in $G_{K/k}$ such that $\pi(\sigma_1\alpha, \sigma\alpha)=0$ for every rational relation $\pi(\alpha, \sigma_2\alpha)=0$, over k. Such Galois hypergroups are all hypergroups $_D$, that is, isomorphic to hypergroups of right cosets of subgroups of finite groups. The author applies his J–H theorem for such hypergroups [Chap. I, Duke Math. J. **6**, 120–140 (1940); these Rev. **1**, 260] to the sequence of hypergroups G_{τ_i}, $\tau_i=Q^{(i-1)}/Q^{(i)}$, $i=1, \cdots, s$, of a generating series $K=Q^{(0)}\supset\cdots\supset Q^{(s)}=k$, of K/k, where k is a p-adic number field, and the Q-chain admits no proper refinement. He employs his generalizations of the Galois theory and Hilbert decomposition theory [Acad. Roy. Belgique. Cl. Sci. Mém. **11**, fasc. 4, 1–110 (1937); Mathematica, Cluj **13**, 72–191 (1937)]. Let p and P be the prime ideals of k and K, respectively, and let $w(\gamma)$ denote the order with respect to P of the integer γ of K. The sets of elements σ of $G_{K/k}$ such that $w(\sigma\gamma-\gamma)-1\geqq 0$, and $w(\sigma\gamma-\gamma)-1>0$, respectively, for all integers γ of K, are the hypergroups T and V of inertia and ramification, respectively, of K/k. They define the fields K_{-1} and K_0 of inertia and ramification, respectively, of K/k. The author proves that the sequence of hypergroups of a generating series of K/k is a composition series of $G_{K/k}$, and hence [Chap. I] is uniquely determined apart from isomorphism and order in the sequence. The latter is also true for general series of K/k which contain both K_{-1} and K_0. A study of the higher ramification numbers of K/k yields necessary and sufficient conditions that a Galois hypergroup be a metagroup.

R. Hull (Vancouver, B. C.).

Weil, André. Sur les fonctions algébriques à corps de constantes fini. C. R. Acad. Sci. Paris **210**, 592–594 (1940). [MF 3031]

The author sketches in this note a proof of Riemann's hypothesis for function fields $K=k(x, y)$ of one variable and genus g over a finite field k of q elements. For the proof, it is necessary to generalize the theory of fixed points of algebraic correspondences C on $\bar{K}=\bar{k}K$, where $\bar{k}$ denotes the algebraic closure of k. The multiplicative group of all $\omega\neq 0$ in $\bar{k}$ is isomorphic to the additive group of all elements mod 1 in the universal ring $\mathfrak{R}$ over the field of rationals with respect to all primes different from the characteristic of k. By means of such a fixed isomorphism and the representation of divisors in $\bar{K}$ by residue classes of functions in $\bar{K}$, the group G of all classes in $\bar{K}$ whose orders are prime to q can be mapped upon the additive group of all $2g$-dimensional vectors with components in $\mathfrak{R}$ mod 1. The automorphism of $\bar{K}|K$ which extends the Frobenius automorphism $\omega\to\omega^q$ induces a linear transformation I on the vector group. The transformation I can be interpreted as a $2g$-dimensional matrix with coefficients in $\mathfrak{R}$. A (m_1, m_2) correspondence C of K induces a homomorphism on G which can be described by a matrix L operating on the representation of G by vectors. The number of fixed points of C is "in general" equal to $m_1+m_2-T_r(L)$. The key to the author's investigation is the fact that the trace $T_r(LL')$ is a positive rational integer, where L' denotes the matrix of the inverse correspondence of C. This result is a generalization of the fundamental relation for the periods of abelian integrals of first kind on an aigebraic Riemann surface. A careful analysis of the ring of correspondences on $\bar{K}$ shows that the latter is a finite hypercomplex order over the ring of rational integers. The actual proof of Riemann's hypothesis consists of the investigation of the correspondence $\Sigma=(x\to x^q,$

$y \to y^q$). The matrix representing this correspondence is equal to I and its characteristic polynomial $|E-uI|$ coincides with the polynomial factor of the zeta function of K. The fundamental inequality $T_r(LL')>0$ applied to the correspondences $a_0+a_1\Sigma+\cdots+a_{2g-1}\Sigma^{2g-1}$, a_i rational integers, proves that all the characteristic roots have $\sqrt{q}$ as absolute value. Finally, the author mentions that his methods prove Artin's series for a finite extension of K to be polynomials. The reviewer hopes that a detailed account of these important results will soon be accessible. *O. F. G. Schilling.*

Zariski, Oscar. Local uniformization on algebraic varieties. Ann. of Math. (2) 41, 852–896 (1940). [MF 3027]

The theorem proved is the following. Let Σ be the field of rational functions on an irreducible r-dimensional algebraic variety V over a field K of characteristic zero, and let W be a sub-variety of V which is the center of an s-dimensional valuation B of Σ over K. Then there is a birational transform V' of V on which the center W' of B is a simple sub-variety of V' and such that the quotient ring of W is contained in the quotient ring of W'. The classical case of this theorem arises when $s=0$ and K is the complex field. The theorem then states that in the space M of all zero-dimensional valuations, topologized in a standard fashion, every point is contained in an open set which corresponds to a portion of V that can be parametrized holomorphically. M is shown to be bicompact, and so V can be covered by a finite set of such parametrizations.

The theorem is first proved when B is a zero-dimensional valuation of rank one, and V is a hypersurface. Here the passage from V to V' can be made by a Cremona transformation of the carrying space, based on the following algorithm of Perron. If $\tau_1, \cdots, \tau_m$ are rationally independent positive real numbers, there exist non-negative integers A_i^j, $i=1, \cdots, m$, $j=1, 2, \cdots$, such that for any $h>0$, $|A_i^h A_i^{h+1} \cdots A_i^{h+m-1}| = \pm 1$, and $\lim_{h\to\infty} A_i^h/A_1^h = \tau_i/\tau_1$. If the residue field K' of B is isomorphic with K, and $v(x_i-c_i) = \tau_i > 0$, $c_i \in K$, $\tau_1, \cdots, \tau_m$ being rationally independent, then the equations $x_i - c_i = X_1^{A_i^h} X_2^{A_i^{h+1}} \cdots X_m^{A_i^{h+m-1}}$, $i=1, \cdots, m$, define a Cremona transformation T^h. In case K' is a finite or infinite algebraic extension of K this definition cannot be used since then the c_i may not belong to K, whereas T^h is to be defined by polynomials over K. The construction of T^h is accomplished here by means of an involved induction on h. Let the equation of the hypersurface be $f(x_1, \cdots, x_{r+1})=0$, and let the center of B be the origin. It is assumed that zero is a root of finite multiplicity $s>0$ of $f(0, \cdots, 0, x_{r+1})=0$. If $s=1$ the origin is a simple point; hence one may proceed by induction on s. A Cremona transformation which is a suitable combination of the T^h replaces s by $s' \leqq s$. If $s'=s$ at each such step, it is shown that the value of $\partial f/\partial x_{r+1}$ must be arbitrarily high. Hence $\partial f/\partial x_{r+1}=0$ on f, which is impossible since f is irreducible. (This method of proof is quite similar to one of the standard ones for algebraic curves.) Here again, the case in which $K'=K$ causes considerable trouble.

The extension of this result to the general case is comparatively simple. If V is not a hypersurface a sufficiently general projection reduces it to one without spoiling the required properties of B. The passage from zero-dimensional valuations of rank one to s-dimensional valuations of rank σ is effected by a simultaneous induction on s and σ. In the first place such a valuation may be reduced to one of dimension zero and rank σ by a suitable transcendental extension of the ground field. Secondly, if B is zero-dimensional of rank $\sigma+1$ it may be obtained by compounding a valuation B_1 of rank σ and some dimension s with a valuation $\bar{B}$ of rank one and dimension zero, defined in the residue field of B_1; B_1 is taken care of by the first part of the induction, and $\bar{B}$ is handled by the special case already proved. Most of the complications in these proofs arise from the use of ground fields which are not algebraically closed. This is unavoidable, because of the necessity of making transcendental extensions in the course of the above induction. Many of the questions involved in the required algebraic and transcendental extensions of K were discussed by the author in a previous paper [Amer. J. Math. 62, 187–221 (1940); these Rev. 1, 102]. *R. J. Walker.*

THEORY OF GROUPS

Miller, G. A. Minimal cross-cut subgroups relative to the product of their orders. Proc. Nat. Acad. Sci. U. S. A. 26, 621–625 (1940). [MF 2897]

Two subgroups H_1 and H_2, of orders h_1 and h_2, of a group G of order g are said to be minimal cross-cut subgroups relative to the product of their orders (m. c. subgroups) if the order k of their cross-cut K is given by $k=h_1h_2/g$. This occurs when every coset of G with respect to H_1 contains k elements of H_2. It is shown (1) that H_1 and H_2 cannot be conjugate subgroups of G; (2) that if each two proper subgroups of G are m. c., then G is the direct product of two groups of prime order; and (3) that if the composite number g^2/h_1h_2 is divisible by just two primes p_1, p_2, the subgroups H_1 and H_2 are m. c. subgroups. *J. S. Frame.*

Miller, G. A. Every two equal order subgroups having only identity in common. Proc. Nat. Acad. Sci. U. S. A. 26, 652–655 (1940). [MF 3144]

Several theorems are proved about a group G which contains a set of subgroups H_i of equal order h such that all the operators of G appear in these subgroups, and each two subgroups have only identity in common. First, it is proved that, if $h=p^k$, then G is of order p^{nk}. Secondly, it is proved that, if G is abelian of order p^m and type 1^m, where $m=nk$, then G contains a set of subgroups of order p^k having the stated property; however, if G contains operators of order p^2, this is not possible for $k>1$. Next it is proved that any abelian group G with the given property must be of order p^m and type 1^m. Finally, if G is non-abelian and has such a set of subgroups H_i, then the Sylow subgroups of each H_i are abelian of type 1^k. *J. S. Frame* (Providence, R. I.).

Kulakoff, A. Einige Bemerkungen zur Arbeit: "Form of the number of the subgroups of a prime power group" von G. A. Miller. Rec. Math. [Mat. Sbornik] N.S. 8 (50), 73–75 (1940). (German. Russian summary) [MF 3190]

The author points out mistakes in the cited paper of G. A. Miller [Proc. Nat. Acad. Sci. U. S. A. 9, 237–238 (1923)] giving examples to show that the reasoning was not valid. He then proves the corresponding theorems with the necessary modifications. The final result still is true, that the number of subgroups of order p^s in a non-cyclic group of order p^m (p an odd prime, $m>1$, $1 \leqq s \leqq m-1$) is congruent to $1+p \pmod{p^2}$. *J. S. Frame.*

Bays, S. et Hsia, Chuin-Ché. Les systèmes imprimitifs dans lesquels se répartissent les combinaisons i à i de m éléments par les substitutions du groupe cyclique de degré m. Comment. Math. Helv. 12, 307–316 (1940). [MF 3059]

In this paper the effect of the cyclic group generated by the substitution $S=(0, 1, 2, \cdots, m-1)$ on the sets of i elements (i-tuples) formed from $0, 1, 2, \cdots, m-1$ is considered. If such an i-tuple is left invariant by S^α but not by S^β, where $\beta<\alpha$, α is called the character of the i-tuple. The main result is the following. Let δ be the greatest common divisor of m and i and d be any divisor of δ. If $C(d)$ denotes the number of combinations of m/d elements taken i/d at a time and $P(d)$ the number of i-tuples of character m/d, then

$$C(d)=\sum P(dd_k),$$

the summation extending over all divisors d_k of δ/d. Applications are made to the particular case $m=96$, $i=12$.

J. Williamson (Baltimore, Md.).

Brauer, Richard and Coxeter, H. S. M. A generalization of theorems of Schönhardt and Mehmke on polytopes. Trans. Roy. Soc. Canada. Sect. III. (3) 34, 29–34 (1940). [MF 3350]

If G is an irreducible finite group of orthogonal transformations of the n-dimensional real space, if $V_1, \cdots, V_k$ are k distinct h-dimensional subspaces which may be obtained from one of them by application of the operations in G, if z is a vector and z_i its orthogonal projection upon V_i, then $khz=n(z_1+\cdots+z_k)$. This theorem is obtained first in proving a generalization of a formula of I. Schur concerning the sums of products of coefficients of pairs of elements g, g^{-1} in G. In case G is absolutely irreducible, a sharper statement is obtained using Schur's original formula.

R. Baer (Urbana, Ill.).

Sigley, Daniel T. k-set groups. Ann. of Math. (2) 41, 767–770 (1940). [MF 3019]

A k-set group is one which contains precisely k complete sets of non-invariant conjugate subgroups. For $k=0$, a k-set group G is Hamiltonian. For $k>0$, the author proves that the order of G is divisible by less than $k+2$ distinct prime factors. An explicit construction is given to show the existence of at least one k-set group for each k. The defining relations for all 1-set groups (three types) and 2-set groups (five types) are given in the two concluding theorems.

J. S. Frame (Providence, R. I.).

Turkin, W. K. On conjugate elements in finite groups. Memorial volume dedicated to D. A. Grave [Sbornik posvjaščenii pamjati D. A. Grave], Moscow, 1940, pp. 254–258. (Russian) [MF 3517]

The author defines a "quasi-normalizer modulo r" of an element A of a finite group $\mathfrak{G}$ as the set $\mathfrak{N}(A, r)$ of all elements X of $\mathfrak{G}$ for which $XAX^{-1}=A^m$ ($m\equiv 1 \pmod r$), where r is some divisor of h, the order of A. Previous results [C. R. (Doklady) Acad. Sci. URSS (N.S.) 3, 359–360 (1936)] for the case $h=p^\alpha$ are here extended to the general h. Three theorems are proved, the first of which may be stated as follows: Let the order h of an element A of a finite group be divisible by $\lambda\mu r$, where λ, μ, r are integers such that r is divisible by $\lambda\mu$. If $\mathfrak{N}(A, r)\neq\mathfrak{N}(A, \lambda r)$, then $\mathfrak{N}(A, \lambda\mu r)$ is a subgroup of order d of the group $\mathfrak{N}(A, \lambda r)$, where d is some divisor of μ.

D. H. Lehmer (Berkeley, Calif.).

Dubuque, P. E. Sur les sous-groupes invariants dans un groupe fini. C. R. (Doklady) Acad. Sci. URSS (N.S.) 24, 104–106 (1939). [MF 2914]

A short report in French listing without proof nine of the twelve theorems of the paper reviewed below.

J. S. Frame (Providence, R. I.).

Dubuque, P. Sur l'invariance des sous-groupes d'un groupe fini. Rec. Math. [Mat. Sbornik] N.S. 7 (49), 285–300 (1940). (Russian. French summary) [MF 2790]

Let A be an element of order p^k of a group $\mathfrak{G}$ and $B=A^{p^i}$. Then the elements N, such that $N^{-1}BN=B\cdot A^{ip^{k-1}}=B^{1+ip^{k-i-1}}$ form a subgroup $\mathfrak{N}_B^{(k-i-1)}$ of $\mathfrak{G}$ called the $(k-1)$st quasi-normalizer of B, which contains as a subgroup the normalizer, or kth quasinormalizer, of B. The author proves twelve theorems which generalize results of a previous paper. The first five theorems assume certain properties of various quasinormalizers, and obtain divisibility properties of the ratio of the orders of two related normalizers. The last seven give criteria that a group be not simple. The condition (I) that in a certain p-group $\mathfrak{P}$ each element conjugate to A^s be of the form A^{ms}, where $m\equiv 1 \pmod p$, is assumed in eight of the theorems. In Theorem 10, for example (Th. 8 of the paper reviewed above), $\mathfrak{P}$ is the Sylow subgroup of the normalizer of the cyclic group $\{A\}$ and $p>2$. Assuming condition (I) and assuming that A does not belong to the commutator group of $\mathfrak{P}$, it is proved that $\mathfrak{G}$ contains an invariant subgroup whose order is a multiple of the largest factor of the order of $\mathfrak{G}$ which is prime to p.

J. S. Frame (Providence, R. I.).

van der Waerden, B. L. Bericht über die Arbeit von H. Fitting, Beiträge zur Theorie der Gruppen endlicher Ordnung. J. Reine Angew. Math. 182, 215 (1940). [MF 3436]

Dietzmann, A. P. Über einige Kriterien der Nichteinfachheit der Gruppen. Rec. Math. [Mat. Sbornik] N.S. 7 (49), 533–538 (1940). (Russian. German summary) [MF 2804]

Without making use of the theory of characters, the author proves several theorems about decomposition of groups. The last is as follows: Let A be an element of prime-power order p^α of a finite group $\mathfrak{G}$, let N be the normalizer of A, $\mathfrak{H}$ a subgroup of $\mathfrak{G}$ of index f containing A with the property that the index in N of $\mathfrak{H}\cap N$ is prime to p, and suppose $\mathfrak{G}=\mathfrak{H}+\mathfrak{H}G_2+\cdots+\mathfrak{H}G_f$. Let h_λ be the order of the class of conjugate elements of $\mathfrak{G}$ containing A^λ. Then if for every λ, $0<\lambda<p^\alpha$, the element A^λ is contained in fewer than $2f/h_\lambda$ of $\mathfrak{H}, G_2^{-1}\mathfrak{H}G_2, \cdots, G_f^{-1}\mathfrak{H}G_f$, and if A is not contained in the commutator-group of $\mathfrak{H}$, it follows that f is not divisible by p and $\mathfrak{G}$ is not simple.

H. Wallman (Chapel Hill, N. C.).

Hall, P. Verbal and marginal subgroups. J. Reine Angew. Math. 182, 156–157 (1940). [MF 3429]

This brief note discusses the possibility of substituting in a number of developments for the commutator $x_1^{-1}x_2^{-1}x_1x_2$ an arbitrary product $f(x_1, \cdots, x_n)$ of the x_i's and their inverses.

F. J. Murray (New York, N. Y.).

Hall, P. The construction of soluble groups. J. Reine Angew. Math. 182, 206–214 (1940). [MF 3435]

The lower nilpotent series for a soluble group G is a sequence $G=L_0\supset L_1\supset\cdots\supset L_n=1$, such that L_{i+1} is the unique minimal self-conjugate subgroup of L_i such that

L_i/L_{i+1} is nilpotent; n is the nilpotent length of G. Let N be a soluble group of length r and D a nilpotent group. The present paper outlines a method for the construction of soluble G's of length $r+1$ and such that $L_1(G)=N$; N and D are partially complementary, that is, $ND=G$ while D does not contain N. This process avoids the construction of duplicates. The nilpotent groups are supposed to be known.
F. J. Murray (New York, N. Y.).

Hall, P. On groups of automorphisms. J. Reine Angew. Math. **182**, 194–204 (1940). [MF 3433]

A "realization" R is a group G and an ordered sequence of subsets of G having a number of explicitly stated relations. If $\mathfrak{S}$ is a set of groups G, a "situation" S is the set of realizations with G in $\mathfrak{S}$. The notion of an automorphism of R is evident. An automorphism of S, however, is defined in terms of the abstract notion of the specified relations. Let $t(R)$ denote the number of automorphisms of R, $t(S)$ those of S. Then a situation S is said to be pure if $1/t(S)=\sum_{R\in S}1/t(R)$. These concepts are discussed with respect to examples both for sets of Abelian and non-Abelian groups. The present paper also outlines the proofs of various formulae involving the number of automorphisms of G, based on the notion of a pure situation. These relations constitute sensitive checks on whether one has obtained all the groups having a specified property.
F. J. Murray (New York, N. Y.).

Specht, Wilhelm. Darstellungstheorie der endlichen Gruppen. J. Reine Angew. Math. **182**, 242–248 (1940). [MF 3440]

This paper describes the theory of the modular representations of a finite group as developed by Brauer and Nesbitt [University of Toronto Studies, Math. Series, no. 4, 1937]. *G. de B. Robinson* (Toronto, Ont.).

Kulakoff, A. A. On regular representation of an abstract group. Memorial volume dedicated to D. A. Grave [Sbornik posvjaščenii pamjati D. A. Grave], Moscow, 1940, pp. 104–109. (Russian) [MF 3508]

Kulakoff, A. Über die reguläre Darstellung einer abstrakten Gruppe. V. Rec. Math. [Mat. Sbornik] N.S. **8 (50)**, 69–72 (1940). (German. Russian summary) [MF 3189]

[The first parts appeared in Rec. Math. [Mat. Sbornik] **2 (44)**, 357–359, 1003–1006 (1937); **3 (45)**, 187–189 (1938); **4 (46)**, 371–373 (1938).] In part V an explicit representation is given for two permutable permutations (P) and (Q) in a regular permutation group. (P) is assumed to be of prime order, but not a power of (Q). The author then gives the form of the general element of an abelian regular group, and shows that if (S) and (T) are any two permutations of a regular group the symbols of a single cycle of (S) either belong to different cycles of (T) or are equally spaced in the cycles of (T) to which they belong. In the latter case the orders of (S) and (T) have a common factor. This theory is applied to the construction of permutations of order p in a group G. *J. S. Frame* (Providence, R. I.).

Fuchs-Rabinowitsch, D. J. Über eine Gruppe mit endlichvielen Erzeugenden und Relationen, die keine isomorphe Darstellung durch Matrizen von endlicher Ordnung zulässt. C. R. (Doklady) Acad. Sci. URSS (N.S.) **27**, 425–426 (1940). [MF 3218]

Proof that the group generated by the elements a, b, x, y subject to the relations $y^2=1$, $xy=yx$, $ax=xya$, $bx=x^2b$ does not admit of any isomorphic representation by means of n-rowed square matrices from a commutative field.
R. Baer (Urbana, Ill.).

Fouxe-Rabinovitch, D. I. Über die Nichteinfachheit einer lokal freien Gruppe. Rec. Math. [Mat. Sbornik] N.S. **7 (49)**, 327–328 (1940). (Russian. German summary) [MF 2795]

In dieser Arbeit betrachten wir eine lokal freie Gruppe, das heisst, eine solche Gruppe, deren beliebige endliche Menge von Elementen eine freie Gruppe erzeugt. Wir beweisen, dass eine lokal freie Gruppe immer einen nichttrivialen Normalteiler enthält. Dazu brauchen wir den folgenden Hilfssatz: Ist ein Element A einer freien Gruppe G vom Einheitselement verschieden, so gehört A nicht zu dem kleinsten Normalteiler von G, der A^2 enthält. Der Beweis des Hilfssatzes stützt sich auf die von Magnus festgestellten Eigenschaften von Reidemeisterschen höheren Kommutatorgruppen. *Author's summary.*

Tschernikow, S. Zur Theorie der unendlichen speziellen Gruppen. Rec. Math. [Mat. Sbornik] N.S. **7 (49)**, 539–548 (1940). (Russian. German summary) [MF 2805]

Continuation of work on infinite special groups [the same Rec. Math. **7 (49)**, 35–64 (1940); cf. these Rev. **2**, 5]. The following are some of the theorems proved. By a p-extension of a group N one understands a group $G\supset N$ in which N is normal and such that the factor group G/N is a p-group. If this factor group is finite G is called a finite p-extension of N. It is shown that every p-extension of a primary Abelian group N of type p^∞ contains N in its centrum. Another theorem is: when an infinite special group contains a finite upper central series, then its centrum is necessarily infinite. The last theorem proved is that the intersection of the centrum of a special group with each of its invariant subgroups (other than the identity) is different from the identity. *M. S. Knebelman* (Pullman, Wash.).

Scholz, Arnold. Zur Abelschen Durchkreuzung. J. Reine Angew. Math. **182**, 216 (1940). [MF 3437]

Simplifications of proofs and explanatory remarks concerning the author's paper: Abelsche Durchkreuzung [Monatsh. Math. Phys. **48**, 340–352 (1939); cf. these Rev. **1**, 104]. *R. Baer* (Urbana, Ill.).

Baer, Reinhold. Abelian groups that are direct summands of every containing abelian group. Bull. Amer. Math. Soc. **46**, 800–806 (1940). [MF 2926]

Let R be a ring which need not be commutative but which has a 1-element. An abelian group G is called an abelian group over R if G admits the elements of R as operators. A homomorphism Φ of a left-ideal M of R into G is defined as a mapping $m\to m^\Phi$ of M upon a subset of G such that $(r_1m_1+r_2m_2)^\Phi=r_1(m_1^\Phi)+r_2(m_2^\Phi)$ for any m_1, m_2 in M and any r_1, r_2 in R. The group G is termed M-complete if, for every homomorphism of M into G, there exists an element v of G such that $m^\Phi=mv$ for every m in M. The group G over R is complete if this condition is satisfied for every left-ideal M of R. It is shown that G is complete if and only if G is a direct summand of every Abelian group over R which contains G. A necessary and sufficient condition for M-completeness is given when M is a principal left-ideal. Finally, it is shown that every Abelian group G over R can be embedded in a complete group G'. This can be done

in such a manner that G' is isomorphic with a subgroup of any complete group containing G. *R. Brauer.*

Kurosch, Alexander. Lokal freie Gruppen. C. R. (Doklady) Acad. Sci. URSS (N.S.) **24**, 99–101 (1939). [MF 2912]

A group G is called locally free if each finite subset of G generates a free group. The free product of an arbitrary set of locally free groups is itself locally free. Each countable locally free group is the union of an increasing sequence of free groups with a finite number of generators. If each of these free groups has n generators, their locally free union will be said to have rank n. In general the rank of a locally free group G is finite and is equal to the least integer n such that each finite subset of G is contained in a free subgroup with n generators. It is shown that the rank of the free product of two locally free groups is equal to the sum of their ranks, and that there are locally free groups of arbitrary power and countable locally free groups of arbitrary finite rank which are free and indecomposable.

J. S. Frame (Providence, R. I.).

Barrett, W. A note on the structure of real semi-simple infinitesimal groups. J. London Math. Soc. **15**, 196–203 (1940). [MF 3138]

For a real form G of a given complex semi-simple infinitesimal group, the conjugate $\bar{\alpha}$ of a root α is again a root, and G is thus associated with an involution $\alpha \to \bar{\alpha}$ of the system of roots. In his thesis [Paris, 1939] the author treated the G by means of these involutions. The involution apparently depends on the choice of the maximal Abelian subgroup by which the notion of roots is defined. The present paper contains the proof that in fact the involution is uniquely determined up to an automorphism, the automorphisms being those linear transformations which leave the system of roots invariant (or, what is the same, those permutations of the roots which do not affect the linear relations holding among them). *H. Weyl.*

Neumer, Walter. Die dreigliedrigen Berührungstransformationsgruppen der Ebene, welche keine Invariante erster Ordnung besitzen. II. J. Reine Angew. Math. **182**, 1–31 (1940). [MF 2376]

[The first part of this paper appeared in J. Reine Angew. Math. **181**, 133–152 (1939); cf. these Rev. **1**, 163.] It classifies the types of subgroups G_3 of a contact transformation similar to the projective G_8 of the plane which do not leave a function $I(x, y, y')$ invariant and their relations to other subgroups, starting from their equations of definition. In the course of this study a connection is established with work of W. Neumer [J. Reine Angew. Math. **179**, 193–203 (1938)] on the differential equation of the conics in the plane. Then follows an investigation of those elements of arc $\xi dx + \nu dy'$ which allow a G_3 of contact transformations [see F. Engel, Jber. Deutsch. Math. Verein. **19** (1910)], with a discussion of the metric induced by such an element of arc. *D. J. Struik* (Cambridge, Mass.).

Drinfeld, G. Sur les opérateurs, permutant les invariants intégraux d'un groupe de transformations continues. Acad. Sci. RSS Ukraine. Rec. Trav. [Zbirnik Prace] Inst. Math. **1940**, no. 4, 157–164 (1940). (Ukrainian. Russian and French summaries) [MF 3234]

In this note G_1 is a one parameter Lie group with symbol $Xf = X_i(\partial f/\partial x^i)$.

$$I = \int A_{\alpha_1 \cdots \alpha_p} dx^{\alpha_1} \cdots dx^{\alpha_p}$$

is an integral invariant of G_1. If

$$B_{\alpha_1 \cdots \alpha_p} = YA_{\alpha_1 \cdots \alpha_p} + \sum_i A_{\alpha_1 \cdots i \cdots \alpha_p} \frac{\partial Y_i}{\partial x^{\alpha_i}}$$

is to define an integral invariant of G_1 for all p, it is necessary and sufficient that $(X, Y)f = \lambda Xf$, where λ is a constant. Hence G_1 is an invariant subgroup of G_2 generated by X and Y. *M. S. Knebelman* (Pullman, Wash.).

v. Neumann, J. and Wigner, E. P. Minimally almost periodic groups. Ann. of Math. (2) **41**, 746–750 (1940). [MF 3016]

A group G is called maximally (or minimally) almost periodic if there exists for every element a distinct from the unit element e (or for no element a) an almost periodic function f on G such that $f(a) \neq f(e)$. The authors prove that the group of linear substitutions: $x = (ux+v)/(wx+z)$, with $uz - vw = 1$ and u, v, w, z rational, is minimally almost periodic; but its subgroup obtained by restricting the u, v, w, z to be integers is maximally almost periodic. The group of substitutions $x = ux + v$, with v and u ($\neq 0$) rational, is not of either extreme kind. While the positive statements are proved by direct construction, the negative statements follow from the lemma that an element a of G which is conjugate in G to some power $(a^n)^m$ of any of its powers a^n, $n > 0$, has the property that $f(a) = f(e)$ for every almost periodic function f on G. *E. R. van Kampen.*

Griffin, Harriet. The abelian quasi-group. Amer. J. Math. **62**, 725–737 (1940). [MF 2873]

The author considers a quasi-group Q of finite order, which is a system such that the product ab of two elements of Q is an element of Q, $ab = ba$, and $ax = b$ has a unique solution x. The following further conditions are imposed. (P_0) Q has a (disjoint) coset decomposition with respect to any subquasi-group. (P_1) A coset is determined by any one of its members. (P_2) The decomposition into cosets is transitive in the sense that, if $Q > A > B$, $Q = A + q_1Q + \cdots + q_nA$, $A = B + a_1B + \cdots + a_rB$, then the decomposition Q/B may be obtained by the substitution of A in terms of B into the decomposition Q/A. Necessary and sufficient conditions for (P_0), (P_1) in the form of weakened associative laws were given by Hausmann and Ore [Amer. J. Math. **59**, 983–1004 (1937)]. The author formulates such a condition for (P_2). The question of the existence of quasi-groups of certain orders and with prescribed properties and the question of the existence of a quotient quasi-group with respect to each subquasi-group are considered. *H. S. Wall.*

Rees, D. On semi-groups. Proc. Cambridge Philos. Soc. **36**, 387–400 (1940). [MF 2918]

By a semigroup is meant a system S closed under a single binary operation satisfying only the associative law. A structure theory is given analogous to that of linear algebras. A subset T of S is called an ideal if $ST \subseteqq S$, $TS \subseteqq S$. The difference semigroup $S - T$ is defined by collapsing T into a single (zero) element, elements outside of T retaining their individuality. Analogues of the "second" and "third" isomorphism theorems of group theory are proved, and it is shown by the Zassenhaus construction that any two normal series of S have isomorphic refinements. S is simple if it contains no proper ideal $\neq (0)$, and completely simple if its idempotent elements satisfy certain further conditions. A finite simple semigroup is completely simple, and is the same thing as a Suschkewitsch Kerngruppe [Math. Ann. **99**, 30–50 (1928)]. A completely simple semigroup is char-

acterized as a "regular matrix semigroup" S over a group G with zero, the elements of which are (finite or infinite) matrices X with at most one component $\neq 0$ in G, the product in S of X and Y being the matrix product XPY, where P is a fixed matrix in G having at least one non-zero component in each row and each column.

A. H. Clifford (Cambridge, Mass.).

Post, Emil L. Polyadic groups. Trans. Amer. Math. Soc. **48**, 208–350 (1940). [MF 2741]

An m-group is constituted by a system S of elements together with an operation $c(s(1), \cdots, s(m))$ subject to the following two rules: if any m of the $m+1$ symbols $s(i)$ in an equation $c(s(1), \cdots, s(m))=s(m+1)$ are elements in S, then the remaining symbol is an element in S and is uniquely determined by this equation; and secondly

$$c[c(s(1), \cdots, s(m)), s(m+1), s(m+2), \cdots, s(2m-1)]=$$
$$c[s(1), c(s(2), \cdots, s(m+1)), s(m+2), \cdots, s(2m-1)]=\cdots.$$

[This definition is essentially due to Dörnte, Math. Z. **29**, 1–19 (1928).] The importance of this concept of polyadic group is due to the fact, discovered by the author, that the theory of polyadic groups is exactly equivalent to the theory of finite cyclic extensions of ordinary groups. More precisely: If X is a coset of the ordinary group G modulo its normal subgroup N such that X generates G/N and such that $m-1$ is the order of X in G/N, then any product of m elements in X is an element in X and the elements in X form an m-group under this operation. Conversely, it is possible to represent every m-group in one and essentially only one way in this fashion as a generating coset of a cyclic quotient group. These facts are basic for the theory evolved by the author. This theory is concerned partly with a generalization of the facts of the classical theory of ordinary groups which is possible to a really astonishing degree, partly with the new possibilities inherent in this more general concept. In this latter context we mention only one of the many problems treated by the author. Every m-group gives rise to a $(2m-1)$-group by introducing the new law of composition:

$$c^*(s(1), \cdots, s(2m-1))$$
$$=c[c(s(1), \cdots, s(m)), s(m+1), \cdots, s(2m-1)].$$

Thus it may be possible to derive a given n-group from an m-group with $m<n$ by means of the indicated method, in particular n-groups may be derived from ordinary groups; and conditions for this reducibility are given. We describe finally how m-groups of transformations may be constructed. Let $T(1), \cdots, T(m-1)$ be $m-1$ disjoint sets of elements such that any two of these sets contain the same [finite or infinite] number of elements. Consider $(m-1)$-tuples of biunivoque correspondences $t(1), \cdots, t(m-1)$ with the property that $t(i)$ maps $T(i)$ upon $T(i+1)$. If $t(j, i)$ are m such $(m-1)$-tuples, then $t(1, i)t(2, i+1) \cdots t(m, i+m-1)$ is such an $(m-1)$-tuple too. Thus it is possible to represent m-groups by such m-groups of permutations or linear substitutions; and the author develops an interesting theory of representations of m-groups on the basis of these concepts.

R. Baer (Urbana, Ill.).

Malcev, A. Über die Einbettung von assoziativen Systemen in Gruppen. II. Rec. Math. [Mat. Sbornik] N.S. **8** (**50**), 251–264 (1940). (Russian. German summary) [MF 3462]

If A is a system with associative multiplication subject to the cancellation-law, then A need not be part of any group. In a previous paper [the same Rec. Math. **6** (**48**), 331–336 (1939); these Rev. **2**, 7] the author has given necessary and sufficient conditions for the possibility of imbedding A into a group which were of the form: certain chains of equations imply the validity of further equations. It is shown now that no finite part of these conditions would be sufficient. If A is part of the group G, and if G is generated by the elements in A, then G is a minimal group containing A. If furthermore every homomorphism of G which induces an isomorphism in A is itself an isomorphism of G, then G is absolutely minimal. If A can be imbedded into a group, then it may be imbedded into an absolutely minimal one; but there may exist two essentially different absolutely minimal groups which contain A.

R. Baer.

Levi, F. W. A problem on rigid motion. Math. Student **8**, 1–10 (1940). [MF 2762]

There exist bounded subsets M of the Euclidean plane which are mapped upon a proper part of themselves by certain rigid motions of the plane. It is possible to determine such sets M in such a way that the outer Lebesgue measure of the set of lost points under a suitable rigid motion is positive. If M is any such set, then its points are distributed on certain concentric circles. The system S of all the rigid motions which map M upon a proper part of M is a semi-group, not containing the identity and not containing more than a continuum of elements. S is commutative and consists of rotations around a fixed point. If conversely T is an abelian semi-group which does not contain the identity and which does not contain more than a continuum of elements, then there exists a bounded subset M of the plane such that T and the system of all the rigid motions that map M upon a proper part are isomorphic.

R. Baer (Urbana, Ill.).

ANALYSIS

Theory of Sets, Theory of Functions of Real Variables

Kurosch, Alexander. Zur Theorie der teilweise geordneten Systeme von endlichen Mengen. Rec. Math. [Mat. Sbornik] N.S. **5** (**47**), 343–346 (1939). (Russian. German summary) [MF 2305]

Let S be a partially ordered collection of finite sets $A_\alpha, A_\beta, A_\gamma, \cdots$ containing for each pair A_α, A_β a third set A_γ with $A_\alpha<A_\gamma$ and $A_\beta<A_\gamma$. To each pair A_α, A_β with $A_\alpha<A_\beta$ let there be assigned a mapping $\pi_{\beta\alpha}$ of A_β on A_α satisfying the condition $\pi_{\gamma\alpha}=\pi_{\gamma\beta}(\pi_{\beta\alpha})$. A set P consisting of elements selected from some of the sets A_α is called a projection set if any two elements of P have a common inverse image in P. The author proves that every projection set is contained in a complete projection set, that is, a projection set containing exactly one element from each set A_α. This is an abstract formulation and proof of a result that had been proved and applied earlier by the author for partially ordered sets of complexes [Compositio Math. **2**, 471–476 (1935)] and by R. Baer for partially ordered sets of finite groups [Amer. J. Math. **59**, 869–888 (1937)].

W. Hurewicz (Chapel Hill, N. C.).

Folley, K. W. A property of a simply ordered set. Bull. Amer. Math. Soc. 46, 940–942 (1940). [MF 3452]

Let M be a simply ordered set containing an everywhere dense subset N such that no subset of M of power less than a certain transfinite cardinal number α is coinitial or cofinal with N, and N contains no pair of neighboring subsets both of power less than α. It is shown that there is a decomposition of M into 2^a disjoint subsets, each of power less than 2^a, such that the union of every 2^a of these subsets has a point in common with every perfect subset of M. This is a generalization of the corresponding result concerning a decomposition of the continuum proved by Sierpinski. It follows that the generalized continuum hypothesis is equivalent to the statement that the set M is the union of disjoint subsets, each of power not greater than α, such that any collection of more than α of them has points in common with every perfect subset of M. The last inequality in the paper should be reversed. *O. Frink* (State College, Pa.).

Gleyzal, André. Order types and structure of orders. Trans. Amer. Math. Soc. 48, 451–466 (1940). [MF 3167]

The paper is a penetrating study of order types and order properties derived from given order properties by means of certain simple operations; it contains a variety of general results on the structure of orders. The reference throughout is to linear orders. The terms type and property, as applied to orders, are used synonymously. If α and β are order properties, the order A is said to be of type $\alpha\beta$ if it can be obtained by substituting for every element of an α-order (=order of type α) an order of type β. This definition leads, by transfinite induction, to the definition of the power α^λ for every order property α and every ordinal λ, and to the limit power, denoted by α^I. Minimal property, for sets in general, is defined as follows. Let α be any set property, Γ a property of set properties; β is a minimla Γ-property implied by α if β has property Γ is implied by α, and is such that, if β' has property Γ and is implied by α, then β' is implied by β. Two minimal Γ-properties implied by α being equivalent in the sense of implication, we may speak of "the" minimal property implied by α; α is iterative if $\alpha\alpha$ is of type α. Theorem: α^I is the minimal iterative property implied by α. Also, α is descending if every segment of an α-order is an α-order; α is extensive if the sum of α-orders over a normal or reverse-normal order is an α-order; α is transitive if it is iterative, descending and extensive. It is proved that, for every order type α, there exists a minimal transitive property implied by α; furthermore, this property may be symbolically represented by α^{DEI} $(=((\alpha^D)^E)^I)$, where D, E and I stand for definite operations that yield properties that are respectively descending, extensive and iterative. For brevity, α^{DEI} is denoted by α^T. The high point of the paper is the following decomposition theorem, which generalizes a well-known theorem of Hausdorff: If A is any order, and α any order property, then either A has property α^T or it is the sum of α^T-orders over an order no proper segment of which has property α^T. This theorem is variously applied. A sampling of other results: The order type scattered is transitive. If α is transitive, and A a given order, A either has property α or is the sum of α-orders over an order no proper segment of which has property α. The minimal, iterative order type which includes normal and reverse-normal as subtypes is the type scattered. If α is any order type, the minimal, extensive supertype of α is $\sigma\alpha$, where σ is the type scattered. The notions dense and scattered are generalized to α-dense and α-scattered, for every order property α, and a number of results are obtained for these generalized concepts. An algebra is indicated for operations on order properties. Thus such equations are proved as α-scattered $=\alpha^{RNT}$, $\alpha^{TT}=\alpha^{TN}$-scattered. An order I_λ, generalizing the order of the positive integers, is defined with the following universality properties: I_λ is scattered, of cardinal $\aleph_\lambda$, and contains all scattered orders of cardinal less than $\aleph_\lambda$. A number of further problems is suggested. *H. Blumberg* (Columbus, Ohio).

Saks, S. Sur un théorème de P. Novikoff. Rec. Math. [Mat. Sbornik] N.S. 7 (49), 373–378 (1940). (French. Russian summary) [MF 2798]

The author states that this article contains extensions of recent results by P. Novikoff published under the title "Sur les ensembles effectivement non dénombrables" (Russian) [Bull. Acad. Sci. URSS [Izvestia Akad. Nauk SSSR] 1939, 35–40]. A function $\phi(x_1, x_2, \cdots, x_n, \cdots)$ which associates with each sequence of points of a metric space E another point of the space E is said to be continuous if $\lim_i x_n^{(i)}=x_n$ implies that

$$\lim_i \phi(x_1^{(i)}, x_2^{(i)}, \cdots, x_n^{(i)}, \cdots)=\phi(x_1, x_2, \cdots, x_n, \cdots).$$

The function ϕ is said to be a function of Novikoff for E if it is continuous and if $\phi(x_1, x_2, \cdots, x_n, \cdots)\neq x_k$ for all k and for each sequence. It is shown that if a function of Novikoff exists for a metric space E then this space contains a perfect set. Also in order that a linear set E possess a function of Novikoff it is necessary and sufficient that it contain a perfect subset each point of which is a boundary point of E. *D. Montgomery* (Northampton, Mass.).

***Piccard, Sophie. Sur les ensembles de distances des ensembles de points d'un espace Euclidien.** Mém. Univ. Neuchâtel, vol. 13. Secrétariat de l'Université, Neuchâtel, 1939. 212 pp.

If A and B are point sets of a Euclidean space, the distance set, or d-set for short, of the pair (A, B), denoted by $D(A, B)$, is the set of all numbers d such that there is a pair of points, one in A and the other in B, whose distance is d. If $A=B$, we set $D(A, B)=D(A)$. The present monograph makes a detailed study of various questions concerning $D(A, B)$, for the most part, concerning $D(A)$ for A linear. A few results in the field under investigation are due to Steinhaus, Sierpiński and Ruziewicz, but after chapter I, the results are almost entirely new.

The first chapter begins with a number of simple geometric interpretations for $D(A)$ and $D(A, B)$, contains some results on the cardinal of $D(A)$, and discusses the d-sets of isolated, open, closed, scattered sets, of G_δ's and of F_σ's. A selection of the results: There exists a planar G_δ whose d-set is not L-measurable (Sierpiński). The d-set of a B-measurable set that has in common with each of its translations at most $\aleph_0$ points is B-measurable. The d-set of an analytic set is analytic (Sierpiński). The d-set of a linear set of positive measure contains an interval with 0 as end point (Steinhaus). If A is a linear set of measure 0, the d-set of the complement of A contains every real number c, $0\leqq c<\infty$ (Sierpiński). The d-set of a linear set of second category contains an interval with 0 as end point. A necessary and sufficient condition that two finite linear sets, each of which has at most one point in common with each of its translations, have identical d-sets is that they be congruent; this proposition is invalid for infinite sets.

The second chapter deals with sets E of a Euclidean space that are congruent with their complements CE. Some of the results: A necessary and sufficient condition that a linear set E be congruent with CE is that the set $D(E)+D(CE)$ does not contain all the positive numbers. If E is a planar set congruent with CE, each of the sets $D(E)$, $D(CE)$, $D(E, CE)$ contains all the positive numbers. If the linear set E is L-measurable and congruent with CE, the set M of positive numbers not contained in $D(E)$ cannot contain an infinite decreasing sequence of numbers. Under the continuum hypothesis, there exists a linear set E which is congruent with CE, non-measurable (L), does not possess the property of Baire and is such that M is denumerable.

The third chapter deals with d-sets of perfect sets, in particular, of perfect sets of the first species (treated by W. H. Young and Mirimanoff). The d-set of such a set is of positive measure, but, if a linear perfect set is of zero measure and has no portion of first species, its d-set may be of positive or of zero measure. There exist linear perfect sets A and B such that $D(A)$ and $D(B)$ are of zero measure, while $D(A, B)$ is of positive measure; also such that $D(A)$ and $D(B)$ are of positive measure, $D(A, B)$ of zero measure. A lengthy study of the d-sets of perfect sets is made from the point of view of systems of numeration.

The fourth chapter considers converse questions, namely, under what conditions a set of real numbers is the d-set of a point set. There exists a denumerable set of non-negative real numbers, including 0, that cannot be the d-set of a point set of a Euclidean space of a finite number of dimensions. For every integer $n \geqq 1$, there exists a finite set of real, non-negative numbers, including 0, which cannot be the d-set of a point set of Euclidean n-space. A necessary and sufficient condition is obtained that a finite set, consisting of two or more non-negative numbers, including 0, be the d-set of a linear set. A necessary and sufficient condition that a denumerable (non-denumerable) set of non-negative numbers be the d-set of a linear set is that it have a denumerable (non-denumerable) number of points in common with a denumerable (non-denumerable) number of its translations. Under the continuum hypothesis, if M is a set of positive numbers of zero measure or of first category, there exists a linear set E such that $D(E)$ contains every positive number not in M. If D is a non-denumerable set of non-negative real numbers, including 0, such that for some positive a there are at most $\aleph_0$ elements of D less than a, then D cannot be the d-set of a point set of Euclidean space. *H. Blumberg* (Columbus, Ohio).

Haupt, Otto, Nöbeling, Georg und Pauc, Christian. Sekanten und Paratingenten in topologischen Abhängigkeitsräumen. J. Reine Angew. Math. **182**, 105–121 (1940). [MF 2819]

In a metric, compact and connected space K a "dependence" of finite sets of (not necessarily different) points $x_1, \cdots, x_q$ is defined. It shall satisfy the following axioms: (1) Every set x_1 is independent. (2) If $x_1 = x_2$, the set x_1, x_2 is dependent. (3) If the set $x_1, \cdots, x_p$ is dependent, the same holds for the set $x_1, \cdots, x_p, x$ for any point x. (4) If the set $x_1, \cdots, x_p$ is independent, and if either of the sets $x_1, \cdots, x_p, y$ and $x_1, \cdots, x_p, z$ is dependent, then the set $x_2, \cdots, x_p, y, z$ is dependent. (5) For any independent set $x_1, \cdots, x_q$ the manifold $L[x_1, \cdots, x_q]$ of the points x for which the set $x_1, \cdots, x_q, x$ is dependent, is closed. In the non-empty space θ certain subsets are called connected. The empty set, every set consisting of a single point, and the sum of any two connected subsets with non-empty crosscut shall be connected. Let $n \geqq 2$ and fixed. To every independent set of n points $x_1, \cdots, x_n$ of K a unique point $\vartheta = \vartheta[x_1, \cdots, x_n]$ of θ shall correspond, and there shall exist to every point ϑ of θ at least one set $x_1, \cdots, x_n$ with $\vartheta = \vartheta[x_1, \cdots, x_n]$. This correspondence shall satisfy the following two conditions: If $L[x_1, x_2, \cdots, x_{n-1}] = L[x_1', x_2, \cdots, x_{n-1}]$, then $\vartheta[x_1, x_2, \cdots, x_n] = \vartheta[x_1', x_2, \cdots, x_n]$. The function $f(x) = \vartheta[x_1, \cdots, x_{n-1}, x]$ shall coordinate a connected subset of θ to every connected subset of $K - L[x_1, \cdots, x_{n-1}]$. Then any two points of θ are contained in a connected subset of θ. In particular, the set θ of the n-dimensional secants of a continuum K in Euclidean m-space is connected. Furthermore, the authors prove the continuity of the [local] paratingents of a locally connected continuum. Their main tool is the following theorem ["Brückensatz"]: Let K be a metric continuum, A a closed subset of K, G_1 and G_2 two closed subsets of A with empty crosscut. Every point of A that is a limit point of a component of $K - A$ shall belong to $G_1 + G_2$. Then the set $K - (G_1 + G_2)$ contains a connected subset, the closure of which has non-empty crosscuts with G_1 and G_2, and which is either contained in A or has no point in common with A. *P. Scherk* (New Haven, Conn.).

Schoenberg, I. J. On metric arcs of vanishing Menger curvature. Ann. of Math. (2) **41**, 715–726 (1940). [MF 3012]

If q, r, s are three pairwise distinct points of a metric space, the real non-negative number

$$\kappa(q, r, s) = [(qr+rs+sq)(qr+rs-sq)(qr-rs+sq) \times (-qr+rs+sq)]^{\frac{1}{2}}/qr \cdot rs \cdot sq$$

measures their curvature, and the space has curvature $\kappa(p)$ at an accumulation point p provided $\kappa(p) = \lim_{q, r, s \to p} \kappa(q, r, s)$. Developing this metric notion of curvature, Menger was led to the theorem that vanishing curvature at each point characterizes metrically line segments among all arcs with the Euclidean four-point property [Math. Ann. **103**, 466–501 (1930)]. The proof was based upon two lemmas: (I) Corresponding to each positive integer n, an arbitrary metric arc B contains $n+1$ points $p_0, p_1, \cdots, p_n$ such that $p_0p_1 = p_1p_2 = \cdots = p_{n-1}p_n$, with p_0, p_n endpoints of B. (II) Let $p_1, p_2, \cdots, p_n$ be n points, each four of which are congruently contained in E_3, and let $p_1', p_2', \cdots, p_n'$ be n points of a plane E_2 such that (a) $p_i'p_{i+1}' = p_ip_{i+1}$ $(i = 1, 2, \cdots, n-1)$, (b) $p_i'p_{i+2}' \leqq p_ip_{i+2}$ $(i = 1, 2, \cdots, n-2)$, (c) the polygon $p_1'p_2' \cdots p_n'p_1'$ is convex. Then $p_1'p_n' \leqq p_1p_n$.

Pointing out that the proof offered for (I) is faulty, Alt and Beer [Ergebnisse Eines Math. Koll. 6, 7 (1935)] established it for arcs in a Euclidean space and hence validated Menger's theorem for Euclidean arcs. The principal contributions of the interesting paper under review are (1) a proof of Lemma (I) for arcs in any semimetric space with continuous metric and (2) a weakening of the hypothesis in the special case of Lemma (II) actually used in the characterization theorem by substituting for the Euclidean four-point property demanded in the lemma the weaker restriction that the set $p_1, p_2, \cdots, p_n$ be Ptolemaic (that is, for each four points p_i, p_j, p_r, p_s of the set, the numbers $p_ip_j \cdot p_rp_s$, $p_ip_r \cdot p_jp_s$, $p_ip_s \cdot p_jp_r$, formed by taking products of "opposite" distances, satisfy the triangle inequality). It follows that line segments are characterized metrically among all metric Ptolemaic arcs by having vanishing curvature at each point. Thus in metric Ptolemaic spaces all "uncurved"

arcs are straight. A discussion of a class of uncurved arcs which are not straight is included in the paper.

L. M. Blumenthal (Columbia, Mo.).

Hornich, Hans. Über eine Zusammensetzung von Mengen. Jber. Deutsch. Math. Verein. **50**, 105–111 (1940). [MF 3069]

Let M_1, M_2 be two non-empty sets in an n-dimensional Euclidean space. Let x_m, y_m be the coordinates of two points x, y, $m=1, 2, \cdots, n$. The set of all points $x+y$ whose coordinates are x_m+y_m will be denoted by (M_1M_2), and called the combination set of M_1, M_2. If M_1, M_2 are both closed and one is bounded their combination is closed. If both given sets are connected their combination set is also connected. Denote the measure of a set M by $f(M)$. If M_1, M_2 and (M_1M_2) are measurable, $f(M_1M_2)\geqq f(M_1)+f(M_2)$. If M and the combination set (MM) are both measurable, then $f(MM)\geqq(n+1)f(M)$. The sets of the type $(MM)=M$ are studied. If they are measurable, their measure is zero. If M_1, M_2 are closed, convex and bounded in the plane, their combination set is also closed, convex and bounded and

$$f(M_1M_2)\leqq((f(M_1))^{\frac{1}{2}}+(f(M_2))^{\frac{1}{2}})^2.$$

If Π_1 is an r-dimensional parallelotope and Π_2 is s-dimensional, then $(\Pi_1\Pi_2)$ is at most $r+s$-dimensional. Denote the r-dimensional volume of a parallelotope by $f_r(\Pi)$. Then, $f_{r+s}(\Pi_1\Pi_2)\leqq f_r(\Pi_1)f_s(\Pi_2)$. Equality results when the parallelotopes are orthogonal. If the dimension of M_1 is r, and of M_2 is s, their combination set is called a translation manifold. Denoting by manifold the topological image of an interval, $f_{r+s}(M_1M_2)\leqq f_r(M_1)f_s(M_2)$. Equality holds for continuously differentiable manifolds if they lie in orthogonal linear parts of R_n. If M_1 is a rectifiable curve of length L, and M_2 is a sphere about the origin of radius d, $f(M_1M_2)\leqq LV_{m-1}(d)+V_m(d)$, where $V_m(d)$ denotes the volume of an m-dimensional sphere of radius d. This theory can be applied to the geometrical theory of absolutely convergent series. For each such series $\sum a_j$ and integer $n>1$, there is a set A_n of the points p of the complex plane of the form $p=\sum a_j\xi^{l_j}$, where $\xi=e^{2\pi i/n}$, and the l_j are arbitrary integers. If the given series is divided into the sum of two partial series $\sum a_j'$, $\sum a_j''$, with corresponding sets A_n', A_n'', then $A_n=(A_n'A_n'')$. Among the many consequences of this relation is the fact that, if each of the sets A_n', A_n'' contain continua, then for $n\geqq 3$ they cannot lie entirely on parallel lines.

E. W. Chittenden (Iowa City, Iowa).

Tornier, Erhard. Neuer Beweis der Äquivalenz der Messtheorien mit verschiedenen Verknüpfungsoperationen. Deutsche Math. **5**, 212–216 (1940). [MF 3117]

A shorter proof of Theorem 2.43 in the author's Wahrscheinlichkeitsrechnung [B. G. Teubner, 1936] is given. The theorem concerns the possibility of replacing a continuous monotone "measure" by one which is additive.

N. Dunford (New Haven, Conn.).

Pospíšil, Bedřich. Über die messbaren Funktionen. Math. Ann. **117**, 327–355 (1940). [MF 3000]

The author develops an abstract-algebraic theory of "measurable functions," based on the following idea. Let f be any measurable real function, A the Boolean ring (in the sense of Stone) of measurable subsets of the domain of f, and for each interval j let $\varphi(j)$ denote the set $a\varepsilon A$ of x with $f(x)\varepsilon j$. Then $\varphi(j_1+j_2)=\varphi(j_1)+\varphi(j_2)$, $\varphi(j_1j_2)=\varphi(j_1)\varphi(j_2)$, and so on. The author considers abstract functions φ with these properties from ordinary intervals j to elements of an abstract Boolean ring A. He defines a character on a set (ring) of such abstract functions to a fixed A in a fashion reminiscent of the notion of the character of an Abelian group. Many results are given, including generalization to functions of N variables.

G. Birkhoff.

Murray, F. J. Nullifying functions. Bull. Amer. Math. Soc. **46**, 459–465 (1940). [MF 2411]

$f(x)$ is nullifying on $I=(0, 1)$ if there is a subset S of I of measure 1 such that the set of numbers $f(x)$, $x\varepsilon S$, is of measure 0. The author shows that there exists a measurable function $f(x)$, defined on I, such that $f+\rho x$ is nullifying for every real ρ. A generalization of this result is indicated.

H. Blumberg (Columbus, Ohio).

Kametani, Shunji. A converse of Lebesgue's density theorem. Proc. Imp. Acad. Tokyo **16**, 350–353 (1940). [MF 3476]

The author establishes the following converse of Lebesgue's density theorem: Let E be a point-set whose lower density is 0 at almost every point of the complementary set of E; then E is measurable. The second theorem extends the first and leads to the third, which states a necessary and sufficient condition that a finite function $f(x)$ defined on a measurable set A in Euclidean space R^m be measurable. Good [Proc. Cambridge Philos. Soc. **36**, 9–13 (1940); cf. these Rev. **1**, 206] has obtained a more precise condition satisfied by a finite measurable function defined on a linear set A. The fourth theorem shows that Good's condition is sufficient for measurability, even when A is a set in R^m, $m>1$.

G. B. Price (Lawrence, Kan.).

Kantorovitch, L. V. On the theory of Stieltjes-Riemann integrals. Leningrad State Univ. Annals [Uchenye Zapiski] Math. Ser. **6**, 52–68 (1939). (Russian) [MF 3289]

A short account of the classical theory of Riemann-Stieltjes integrals and some of their applications.

J. W. Tukey (Princeton, N. J.).

Bosanquet, L. S. A property of Cesàro-Perron integrals. Proc. Edinburgh Math. Soc. (2) **6**, 160–165 (1940). [MF 2849]

The Cesàro-Perron integral of order μ, $C_\mu P$-integral, $\mu\geqq 0$, was devised by J. C. Burkill [Proc. London Math. Soc. (2) **34**, 314–322 (1932); **39**, 541–552 (1935); J. London Math. Soc. **11**, 43–48, 220–226 (1936)]. The present paper is a step forward in this theory. Let $f(t)$ be $C_\mu P$-integrable in $(a-\eta, a+\eta)$, where $\mu=\max(\lambda-1, 0)$, $\lambda>0$. Let

$$C_\lambda(f, a, a+h)=\frac{\lambda}{h}\int_0^h\left(1-\frac{u}{h}\right)^{\lambda-1}f(a+u)du,$$

where the integral is in the $C_\mu P$ sense. If, as $h\to 0$, $C_\lambda(f, a, a+h)\to l$, then l is the C_λ-limit of $f(t)$ as $t\to a$. If $l=f(a)$, $f(t)$ is C_λ-continuous at a. If $\lambda\geqq 0$ and $f(t)$ is $C_\lambda P$-integrable to $F(\alpha, \beta)$ for $a<\alpha<\beta<b$, and C_λ-limit of $F(\alpha, \beta)$ exists as $\alpha\to a+0$, $\beta\to b-0$, then $f(t)$ is $C_\lambda P$-integrable on (a, b) with this limit as its value. The proof consists mainly in showing that, if $F(x)=C_\lambda\text{-lim } F(\alpha, x)$, $\alpha\to a+0$, $F(a)=0$, $F(b)=C_\lambda\text{-lim } F(x)$, $x\to b-0$, then there exists $M(x)$ with $M(a)=0$, $M(x)$ C_λ-continuous, $0\leqq M(x)-F(x)<\epsilon$, $C_\lambda D_*M(x)\geqq f(x)$, $C_\lambda D_*M(x)>-\infty$, where

$$C_\lambda D_*M=\varliminf_{h\to 0}\{C_\lambda(M, x, x+h)-M(x)\}/h(\lambda+1).$$

R. L. Jeffery (Wolfville, N. S.).

Tolstoff, G. La méthode de Perron pour l'intégrale de Denjoy. Rec. Math. [Mat. Sbornik] N.S. **8** (**50**), 149–168 (1940). (French. Russian summary) [MF 3196]

The author here completes his paper under the same title [C. R. (Doklady) Acad. Sci. URSS (N.S.) **25**, 470–472 (1939); cf. these Rev. **1**, 305]. He shows that the form of the Perron integral there introduced is equivalent to the general Denjoy integral. A lemma fundamental to this theorem is: Whatever be the set E of zero measure on (a, b), there exists a function $\lambda(x)$, $\lambda(a)=0$, continuous, non-decreasing, everywhere derivable, and such that $\lambda'(x)=+\infty$ for each x in E. The proof of this lemma occupies the greater part of the paper. *R. L. Jeffery* (Wolfville, N. S.).

Guareschi, Giacinto. Sulla rappresentabilità regolare di una varietà di Jordan a punti tutti semplici e spazio tangente variabile con continuità. Boll. Un. Mat. Ital. (2) **2**, 107–109 (1940). [MF 2967]

Let M_n be a variety of Jordan all points of which are simple (in the sense of Severi). Let also the tangent plane at any point P of M_n be a continuous function of P. Then it is proved that M_n can be decomposed into a finite set of cells such that each cell possesses a regular representation (by means of single-valued functions having continuous derivatives). *O. Zariski* (Cambridge, Mass.).

Guareschi, Giacinto. Sull'eliminazione del principio di Zermelo dalla dimostrazione del criterio di Severi sugli estremi relativi delle funzioni. Boll. Un. Mat. Ital. (2) **2**, 110–112 (1940). [MF 2968]

Severi [Ann. Mat. Pura Appl. **13**, 1–35 (1934)] has proved the theorem: "Let P_0 be a limit point of the plane closed set I, and let $f(P)$ be a continuous function defined on I; let $f(P_0)$ be a (relative) extremum of the values taken by $f(P)$ when P approaches P_0 in such a way that the halfray P_0P tends to a well determined halftangent of I at P_0. Then $f(P_0)$ is a (relative) extremum of $f(P)$." A particular case of Zermelo's Axiom was used by Severi in the proof. In the present paper the author gives a new proof of the theorem in which no use at all is made of Zermelo's Axiom. *A. González Domínguez* (Buenos Aires).

Agnew, Ralph Palmer. On continuity and periodicity of measurable functions. Ann. of Math. (2) **41**, 727–733 (1940). [MF 3013]

Let the complex valued function $x(t)$ be measurable on $-\infty<t<\infty$. There exist positive numbers $H_1>H_2>\cdots$ converging to zero such that, for any sequence h_n with $|h_n|<H_n$, $x(t+h_n)\to x(t)$ for almost all t. That H_n cannot be any sequence tending to zero is shown by the following: Let $\varphi(t)$ be the characteristic function of the non-dense set W. There exists $h_n>0$, $h_n\to 0$ and $\limsup|\varphi(t+h_n)-\varphi(t)|=1$ for all t in W. A measurable function $x(t)$ is essentially periodic with essential period h if $x(t+h)-x(t)=0$ for almost all t. If $x(t)$ is essentially periodic and not a constant for almost all t, then $x(t)$ has a least positive essential period h, and each essential period is an integer multiple of h. *R. L. Jeffery* (Wolfville, N. S.).

Functional Equations and Operational Calculus

Moore, M. G. Fundamental systems of solutions for linear difference equations. Duke Math. J. **6**, 652–657 (1940). [MF 2721]

The difference equation $\sum_{\mu=1}^{N}c_\mu f(x+\alpha_\mu)=0$ in which the c_μ and α_μ are complex constants admits of solutions $x^s\exp(t_m x)$ if $t=t_m$ is a root of certain multiplicity of the exponential equation $\sum_{\mu=1}^{N}c_\mu\exp(\alpha_\mu t)=0$. Referring to his paper in Amer. J. Math. **62**, 83–90 (1940) [these Rev. **1**, 141], and by means of further analysis of contour integrals of the type there considered, the author derives conditions under which an integrable $f(x)$ may be approximated by linear combinations of solutions $x^s\exp(t_m x)$ of the given difference equation. The conditions are too intricate to be profitably restated here. *R. E. Langer* (Madison, Wis.).

Lancaster, Otis E. Sequences defined by non-linear algebraic difference equations. Ann. of Math. (2) **42**, 251–280 (1941). [MF 3683]

In the background of this work are certain ideas of G. D. Birkhoff, as well as some investigations of S. Lattès, J. Horn, O. Perron and E. Schröder. The author considers the algebraic difference equation

$$(1)\qquad \phi(s_{n+m}, s_{n+m-1}, \cdots, s_n, n)=0, \quad s_{n+i}=y(n+i),$$

where ϕ is a polynomial with rational coefficients in the arguments; (1) defines a sequence $s_n, s_{n+1}, \cdots, s_{n+m}, \cdots$, designated by (s_n), when values of y are given for $n, n+1, \cdots, n+m-1$. (L) is the linear part of ϕ equated to zero. Some of the typical results are as follows: If in ϕ a coefficient of a linear term is of as high degree in n as the coefficients of the non-linear terms, and if there are solutions of (L) tending to zero (as $n\to+\infty$) and if the c.e. (characteristic equation) of (L) has finite roots, $\neq 0$, $\neq 1$ (in absolute value), then there are sequences (s_n) tending to zero. If in (1) the terms are of at least the first degree in the s_{n+i}, then (1) defines sequences tending to zero, if all the roots of the c.e. of (L) are less than unity in absolute value. For a given integer k and an algebraic number α, there exist equations of order m defining rational sequences tending to α with an order of rate of convergence k. The author also investigates questions regarding the minimum degree for a first order equation whose sequences tend to $\beta^{1/p}$ (β rational; integer $p>0$) with a given order of convergence, as well as questions of approximation of $\beta^{1/p}$. *W. J. Trjitzinsky.*

Lancaster, Otis E. Orthogonal polynomials defined by difference equations. Amer. J. Math. **63**, 185–207 (1941). [MF 3642]

The adjoint equation of the even-ordered difference equation

$$(1)\qquad L[y]\equiv\sum_{i=0}^{2n}P_i(x)y(x+\overline{2n-i}h)=0$$

is defined to be the equation

$$(2)\qquad \bar{L}[v]\equiv\sum_{i=0}^{2n}P_{2n-i}(x+\overline{n-i}h)v(x+\overline{2n-i}h)=0.$$

It has the property that $v(x+nh)L[y]-y(x+nh)\bar{L}[v]$ is the h-difference ($\Delta_h f(x)=[f(x+h)-f(x)]/h$) of a bilinear function of y and v, in analogy with differential equation theory. The adjoint of (2) is (1). If $L=\bar{L}$, the equation is self-adjoint. A second order equation is always self-adjoint on multiplying through by a suitable factor. Two functions $\phi(x)$, $\psi(x)$ are defined to be S-orthogonal on an interval (μ, ν), relative to a weight function $g(x)$, if $S_\mu^\nu g(x)\phi(x)\psi(x)\Delta_h x=0$, where, if $q(x)$ is such that $f(x)=\Delta_h q(x)$, then $S_\mu^\nu f(x)\Delta_h x=q(x)\big|_\mu^\nu q(\nu)-q(\mu)$. (The S-operator is related to the principal Nörlund sum.)

The paper deals mostly with the second order equation

$$(3)\qquad (ax^2+bx+c)\Delta_h^2 y(x)+(dx+e)\Delta_h y(x)+\lambda y(x+h)=0,$$

which, in self-adjoint form, is

$$(4) \qquad \Delta_h(w(x)\Delta_h y(x))+\lambda s(x)y(x+h)=0,$$

where $w(x)$ and $s(x)$ are determined. (3) is shown to have a polynomial solution $y_m(x)$ of degree m if $\lambda=\lambda_m=-am(m-1)-dm$ $(m=0, 1, 2, \cdots)$; and an explicit representation is found for y_m, using (4):

$$(5) \quad y_m(x)=\exp\left\{S\frac{1}{h}\log\left[\frac{ax^2+bx+c-h(dx+f)}{a(x-h)^2+b(x-h)+c}\right]\Delta_h x\right\}$$

$$\times\Delta_h^m\left[\exp\left\{S\frac{1}{h}\log\left[\frac{a(x-h)^2+b(x-h)+c}{a(x-mh)^2+(b-dh)(x-mh)+c-hf}\right]\Delta_h x\right\}\right].$$

(5) is reduced to four particular forms, according to the following cases: $a\neq 0$; $a=0$, $b\neq 0$, $b-dh\neq 0$; $a=0$, $b=0$, $d\neq 0$, $c\neq 0$; $a=0$, $b\neq 0$, $b-dh=0$, $c-hf\neq 0$. In the last three cases conditions are found when the polynomials are S-orthogonal (interval and weight function determined), and also when the zeros of $y_n(x)$ are real, and when they separate those of $y_{n-1}(x)$. The solutions (5) are shown to satisfy a recurrence relation

$$(6) \quad A(n)y_{n+2}(x)+[B(n)x+C(n)]y_{n+1}(x)+D_n y_n(x)=0,$$

the coefficients $A_n, \cdots, D_n$ being determined explicitly. They are now shown to include sets considered by other writers (analogues of the polynomials of Legendre, Hermite, etc.), as well as a Jacobi polynomial analogue; and on letting $h\to 0$, these become the familiar polynomials of Legendre, etc., that satisfy the differential equation to which equation (3) reduces. *I. M. Sheffer* (State College, Pa.).

Hornstein, M. Einige Bemerkungen über lineare Differenzengleichungen zweiter Ordnung und über Kettenbrüche. Rec. Math. [Mat. Sbornik] N.S. **5** (47), 269–288 (1939). (Russian. German summary) [MF 2298]

This paper is concerned with that part of the theory of difference equations which has connection with the theory of continued fractions [H. Poincaré, S. Pincherle, O. Perron]. Under consideration is the equation

$$D_{\nu+2}+a_\nu D_{\nu+1}+b_\nu D_\nu=0, \qquad \nu=0, 1, 2, \cdots,$$

where $\lim a_\nu=a$, $\lim b_\nu=b$ (all $b_\nu\neq 0$). It is assumed that the roots ρ_1, ρ_2 of $\rho^2+a\rho+b=0$ have unequal absolute values. As is known, there exist two distinct solutions (forming a "fundamental system") so that

$$\lim_\nu [D_{\nu+1}^{(i)}/D_\nu^{(i)}]=\rho_i, \qquad i=1, 2.$$

The author gives effective means (presenting an improvement over Pincherle's method) for determination of the initial values of these solutions; for this purpose use is made of certain continued fractions. Also, applications are given to the study of convergence of such fractions.

W. J. Trjitzinsky (Princeton, N. J.).

Birkhoff, George D. Note on linear difference and differential equations. Proc. Nat. Acad. Sci. U. S. A. **27**, 65–67 (1941). [MF 3655]

The author remarks that his earlier work [Acta Math. **54**, 205–246 (1930); G. D. Birkhoff and Trjitzinsky, ibid. **60**, 1–89 (1933)], in which a general analytic theory of linear difference equations was based upon the construction of a full complement of formal solutions, yielded a methodological pattern upon which a similar development for differential equations was subsequently carried through. He asserts in this note that the methodological relationship is in fact so close that the theory for differential equations falls out as an immediate corollary from the theory of difference equations. The assertion is substantiated in part by a consideration of the nth order matrix differential equation $Y'(x)=A(x)Y(x)$, in which the elements of $A(x)$ are analytic at $x=\infty$. A matrix solution of this equation satisfies a difference equation $Y(x+1)=M(x)Y(x)$, with $M(x)$ analytic at ∞. In the case considered, that is, when the roots of the equation $|\rho\delta_{ij}-m_{ij}(\infty)|=0$ are distinct, the difference equation admits of a formal solution $S(x)=(\rho_i{}^x x^{r_i} s_{i,j}(x))$ with the $s_{i,j}(x)$ power series in $1/x$. It is shown that the differential equation therefore has solutions which in the sectors $K\pi-\pi/2-\epsilon\leqq\arg x\leqq K\pi+\pi/2+\epsilon$, $K=0, 1$, are asymptotic to $S(x)$. *R. E. Langer* (Madison, Wis.).

Heins, Albert E. On the solution of linear difference differential equations. J. Math. Phys. Mass. Inst. Tech. **19**, 153–157 (1940). [MF 1780]

The use of the Laplace transformation in solving difference-differential equations is illustrated by carrying out the solution of the equation

$$\frac{d^2y_\lambda}{dt^2}+2y_\lambda-y_{\lambda-1}-y_{\lambda+1}=0$$

under the conditions that $y_0(t)=y_{2n+1}(t)=0$, and that the values of y_λ and dy_λ/dt are prescribed at $t=0$.

R. V. Churchill (Ann Arbor, Mich.).

Jackson, F. H. The q^δ equations whose solutions are products of solutions of q^δ equations of lower order. Quart. J. Math., Oxford Ser. **11**, 1–17 (1940). [MF 1862]

Notations: $a_0=a(x)$, $a_n=a(q^n x)$, and similarly for other letters; let q^δ be an operator which operating on a function $\phi(x)$ converts it into $\phi(qx)$.

Theorem A. Let $\{q^{2\delta}+a_0q^\delta+b_0\}y=0$, $\{q^{2\delta}+c_0q^\delta+d_0\}\eta=0$ be two q^δ-equations satisfied by $y=\phi(x)$ and $\eta=\psi(x)$, respectively; let $b_2c_2d_1/c_0=d_2a_2b_1/a_0=\lambda$; then the q^δ-equation of fourth order

$$\{q^{4\delta}-a_2c_2q^{2\delta}+(c_1c_2b_2+a_1a_2d_2-b_2d_2-\lambda)q^{2\delta} -\lambda a_0c_0q^\delta+\lambda b_0d_0\}Y=0$$

is satisfied by $Y=\phi(x)\psi(x)$. Under certain conditions the product of solutions of two q^δ-equations of the second order satisfies a q^δ-equation of the third (not fourth) order. Theorem B. If $\{q^{4\delta}+a_0q^{2\delta}+b_0\}y=0$ be satisfied by $y=\phi(x)$ then $\{q^{4\delta}+a_1q^{2\delta}-a_0b_1q^\delta-bb_1\}Y=0$ will be satisfied by $Y=\phi(x)\phi(qx)$. Theorem C. If $\{q^{n\delta}+A_0q^{(n-1)\delta}+B_0q^{(n-2)\delta}+\cdots+K_0\}y=0$ be satisfied by $y=\phi(x)$ and $\{q^\delta-c_0\}\eta=0$ by $\eta=\psi(x)$, then $\{q^{n\delta}+A_0c_{n-1}q^{(n-1)\delta}+B_0c_{n-1}c_{n-2}q^{(n-2)\delta}+\cdots+K_0c_{n-1}\cdots c_1\}Y=0$ will be satisfied by $Y=\phi(x)\psi(x)$. By suitable choice of c_0 in the auxiliary equation of first order, it is often possible to reduce linear q^δ-equations to q-hypergeometric forms.

The second and longer part of the paper brings a series of interesting examples for, and applications of, the general theorems of the first part. The actual formulae are too numerous to be given here. It should be mentioned, however, that the paper contains among other results the twenty-four integrals of the q-hypergeometric equation of the second order. Fourteen solutions are written out fully and the remaining ten are indicated. *A. Erdélyi.*

Mahler, Kurt. On a special functional equation. J. London Math. Soc. **15**, 115–123 (1940). [MF 2521]

Asymptotic formulae for solutions of the functional

equation

$$(1) \qquad \frac{f(z+\omega)-f(z)}{\omega}=f(qz), \qquad \omega\neq 0;\ 0<q<1.$$

First a special solution $F(z, \omega, q)$ is constructed with the help of the transformation

$$f(z)=\int_{-\infty}^{+\infty} u(x)e^{z\cdot q^{xi}}dx,$$

which transforms (1) into a simple difference equation for u. For $\omega\to 0$ this special solution becomes

$$F(z, q)=\sum_{n=0}^{\infty}\frac{q^{\frac{1}{2}n(n-1)}z^n}{n!}.$$

The author's result is that for every solution $f(z)$ of (1), which is greater than a positive constant, and is bounded in every finite interval, $f(z)=e^{O(1)}F(z)$ for large z; in particular

$$\log f(z)\sim(\log z)^2/(2\log(1/q)).$$

From this result, an approximate formula for the number of solutions of $h=h_0+rh_1+r^2h_2+\cdots$, $r\geqq 2$ a constant integer, in nonnegative integers $h_0, h_1, h_2, \cdots$ is derived, when h is large. *F. John* (Lexington, Ky.).

Van der Lyn, G. Sur l'équation fonctionnelle $f(x+y)+f(x-y)=2f(x)\phi(y)$. Mathematica, Cluj **16**, 91–96 (1940). [MF 2489]

The author considers the equation $f(x+y)+f(x-y)=2f(x)\phi(y)$, where f and ϕ are complex functions of a complex variable. It is shown that if f and ϕ satisfy the Baire condition and are finite on a residual set then the only nontrivial solutions are $f(x)=ae^{\alpha x}+\beta e^{-\alpha x}$, $2\phi(x)=e^{\alpha x}+e^{-\alpha x}$. The function f is shown to be bounded for bounded x, then continuous, and finally it is shown to be analytic. The equations are then easily solved. *A. C. Schaeffer.*

de Mira Fernandes, A. A multiplication theorem. Portugaliae Math. **1**, 340–342 (1940). (Portuguese) [MF 2894]

Let $X(t)$ be a matrix whose elements are functions of the scalar variable t (not defined for $t=0$) which satisfies the functional equation $X(st)=X(s)+X(t)$. The author shows that if $X(t)$ is continuous at $t=1$, then $X(t)$ satisfies the differential equation $BX'=X$, where B is the scalar matrix $t\log(t)\cdot E$. *L. M. Blumenthal* (Columbia, Mo.).

Silberstein, Ludwik. Solution of the equation $f'(x)=f(1/x)$. Philos. Mag. (7) **30**, 185–186 (1940). [MF 3028]

To satisfy the equation mentioned in the title the author puts $f(x)=a(x^m+\lambda x^n)$, and finds the conditions $\lambda=m$, $n=1-m$ and $m^2-m+1=0$. *W. Feller.*

Silberstein, Ludwik. Differential operators acting as integrators. Philos. Mag. (7) **29**, 586–600 (1940). [MF 2474]

It is shown that the functional, or averaging operator, which yields the arithmetic-mean value of a single-valued scalar field taken over all points in a Euclidean space of 1, 2 or 3 dimensions which are equidistant from a chosen point, can in each case be expressed as a series of even powers of the Laplacian differential operator. The averaging operators are applied at the chosen point, and, in particular, can be expressed in compact form by means of a hyperbolic cosine, first order Bessel function of the first kind with imaginary argument, and a hyperbolic sine in 1, 2 and 3 dimensions, respectively. Of course, the operator is an idemfactor for harmonic-function operands. The integral operation of averaging is replaced by a linear combination of repeated differentiation operations. This explains the title. Eight examples are given in which this replacement of one type of operation by another leads to a simplification. From one of these the author concludes that a radial Bessel field is the next most simple field to a harmonic field. *J. L. Barnes* (Medford, Mass.).

***Ertel, Hans. Elemente der Operatorenrechnung mit geophysikalischen Anwendungen.** J. Springer, Berlin, 1940. vi+133 pp. 7.50 RM.

***Wagner, Karl Willy. Operatorenrechnung nebst Anwendungen in Physik und Technik.** J. A. Barth, Leipzig, 1940. xiv+448 pp. 27.60 RM.

Jeffreys, Harold and Dalzell, D. P. On the Heaviside operational calculus. Proc. Cambridge Philos. Soc. **36**, 267–282 (1940). [MF 2528]

Except for §3, which was written by the second named author, the paper is by the first author. The paragraphs through 2.3 essentially constitute addenda to the first author's tract "Operational Methods in Mathematical Physics" [Cambridge, 1927 and 1931]. The historical references here in each case ignore important earlier literature on the Laplace transformation and its inversion. Partial proofs are supplied for several familiar theorems suggested in the tract, which concern ordinary linear differential equations.

The remainder of the paper is concerned with the difficulty of rigorously extending the methods used in treating systems of ordinary differential equations to partial differential equations. No mention is made of the work of W. Mächler [Comment. Math. Helv. **5**, 256–304 (1933)]. Retreating from the mathematical difficulties, refuge is sought in the old idea that partial differential equations are merely approximate descriptions for an ultimately granular physical world. The interest then centers on the problem of determining the conditions which a discrete system must satisfy for the operational method to be valid. A conclusion is deduced for the mean-square case; but the conditions used in the derivation are not stated. Although an appeal to physical theory is used as the basis for avoiding the usual mathematical description of a continuous system and its difficulties, nevertheless, the conclusion, as stated, is that the discrete systems considered shall not tend to infinite instability. No hint is given as to the physical interpretation of the phrase "infinite instability." *J. L. Barnes.*

Pipes, Louis A. A matrix generalization of Heaviside's expansion theorem. J. Franklin Inst. **230**, 483–499 (1940). [MF 2868]

The well-known solution of the integro-differential equation of an electric circuit with linear bilateral inductance, resistance and elastance elements under arbitrary initial conditions is given by formal use of the p-multiplied Laplace transformation. By employing matrices the solution is then formally generalized in the usual way to cover the case of an electric network with n loops each containing all three linear bilateral elements. A formal treatment of impulses is included. The method of application of synthetic substitu-

tion to evaluate polynomials for complex values of argument is quoted. Although generality is claimed, the treatment does not take into account cases in which some or all elements of a given type are absent from the network. From a practical point of view, the chosen method of solution involving a convolution integral is, in all but the simplest problems, the most difficult to evaluate of the methods available. *J. L. Barnes* (Medford, Mass.).

GEOMETRY

***Robinson, Gilbert de B. The Foundations of Geometry.** Mathematical Expositions, no. 1. University of Toronto Press, Toronto, Ont., 1940. xi+167 pp. $2.00.

Contents: I. Philosophical preliminaries. II. Projective geometry and Desargues' theorem (axioms; a finite geometry; Desargues' theorem; duality; the fourth harmonic point; a harmonic sequence and Fano's axiom). III. Projective geometry and Pappus' theorem (related ranges of points; the reduction of a projectivity to two perspectivities; Pappus' theorem; the fundamental theorem on projectivities; conics). IV. Affine and Euclidean geometry. V. Axioms of Euclidean geometry. Ideal elements. VI. Numbers (integral, rational, real and complex numbers; rings and fields; finite fields). VII. Coordinate systems. VIII. Order and continuity (ordered fields; order in projective, affine and Euclidean geometry; axioms of order and of continuity; von Staudt's continuity proof of the fundamental theorem; Desargues' theorem in the plane; consistency and categoricalness). IX. Correspondences and imaginary elements in geometry (projectivities; involutions; collineations; correlations).

***Vahlen, Theodor. Abstrakte Geometrie. Untersuchungen über die Grundlagen der Euklidischen und nicht-Euklidischen Geometrie.** Zweite, neubearbeitete Auflage. Zweites Beiheft zu Deutsche Mathematik. S. Hirzel, Leipzig, 1940. 224 pp. RM. 8.

Save for a few minor changes this constitutes a reissue of the first edition which appeared in 1905, which has been for some years out of print, and is not easily available. The changes are practically confined to mention of symbols for projective and perspective relations, introduction of a few footnote references to recent work, use throughout of German black-letter type in place of Latin type, and a second preface reflecting a new ideology. The rigorous, logical form of presentation is now no longer novel. The work falls into five parts: I. The foundations of arithmetic; II. Projective geometry (incidence relations); III. Projective geometry (order relations); IV. Affine geometry; V. Metric geometry. There is considerable emphasis upon non-Euclidean geometry throughout. Both analytic and synthetic methods are employed. The number of postulates is disturbingly large. Despite the announced intention of introducing no postulate without proof of its independence, such proof seems never to be forthcoming in the immediate context. The reader may note with regret that essentially only linear relations are considered. The circle, sphere, conic section (save for brief use of the non-Euclidean absolute) are not mentioned. *A. A. Bennett* (Providence, R. I.).

Toepken, Heinrich. Zur absoluten Geometrie. Deutsche Math. **5**, 85–94 (1940). [MF 2810]

Employing a foundation of absolute plane geometry due to Hjelmslev [Danske Vid. Selsk. Math.-Fys. Medd. **8**, Abh. 11; **10**, Abh. 1 (1929)] in which reflexion in a line, as a special case of the primitive notion of "displacement," plays a central rôle, the author proves the following theorem: Let $a=AB$ and $d=CD$, $b=BC$ and $e=AD$, $c=CA$ and $f=BD$ be the three pairs of opposite sides of the quadrangle whose vertices are the ordinary points A, B, C, D, and denote by a', b', c', $\cdots$, f' reflexions in the perpendiculars from an ordinary point P to the lines a, b, c, $\cdots$, f, respectively. If $b'c'=f'e'$, then $a'f'=c'd'$ and $a'b'=e'd'$. This theorem (a kind of dual of the Satz der Gegenpaarung of Hjelmslev) is used to prove the so-called ideal case of Pascal's Theorem. Since the theorem of Desargues is a consequence of Pascal's Theorem, the special case of the latter proved by the author gives rise to the following special case of the former: Let A_1, A_2, A_3 and B_1, B_2, B_3 be two triangles perspective with respect to the ordinary point 0 and let the two pairs of lines A_1A_2 and B_1B_2, A_1A_3 and B_1B_3 each have common perpendiculars intersecting in an ordinary point P. Then the lines A_2A_3 and B_2B_3 have a common perpendicular that goes through P. The paper concludes with an introduction of ideal lines and the proof of the general Pascal Theorem for the extended system of ordinary and ideal points and lines. *L. M. Blumenthal.*

Steck, Max. Über die Äquivalenz zweier Geometrien mit unvollständiger Anordnung. Monatsh. Math. Phys. **49**, 209–212 (1940). [MF 2997]

This note is devoted to showing that the geometry with incomplete order arising from the Bottema postulate that the center of an ellipse is inside the ellipse [Math. Ann. **117**, 17–26 (1939); these Rev. **1**, 165] is equivalent to the geometry obtained by replacing this axiom with the author's weak E.P. Axiom [Math. Ann. **117**, 195–196 (1940); these Rev. **1**, 260] which asserts that corresponding to any non-degenerate conic there exists at least one point such that all lines on this point have two distinct points in common with the conic. The geometries are real projective with linear order relation but without the Pasch axiom holding.

L. M. Blumenthal (Columbia, Mo.).

Steck, Max. Beweis des zweiten Fanoschen Axioms F_2 im Rahmen der Axiomatik der synthetischen Geometrie. Monatsh. Math. Phys. **49**, 213–218 (1940). [MF 2998]

Fano's axiom F_2 asserts that if a net of rationality exists each such net does not contain only a finite number of points. The paper under review gives a proof of this assertion in the framework of Liebmann's "Synthetischer Geometrie" [Leipzig-Berlin, 1934]. It is noted that the E.P. Axiom used in the proof cannot be replaced by the weak E.P. Axiom exploited by the author in previous papers [e.g., Math. Ann. **117**, 195–196 (1940); these Rev. **1**, 260].

L. M. Blumenthal (Columbia, Mo.).

Rachevsky, P. Sur l'unicité de la géométrie projective dans le plan. Rec. Math. [Mat. Sbornik] N.S. **8** (**50**), 107–120 (1940). (French. Russian summary) [MF 3193]

The axiom of Desargues requires that, if all the incidences arising in the Desargues configuration (10_3) are satisfied except one, then this one shall take place automatically. The adjunction of this demand to the ordinary axioms of belonging, order and continuity establishes the "true" projective line. The axiom of Pappus, which is also of this

nature, may be used with the same effect. The author considers the possibility of constructing a new projective geometry by taking instead of the axiom of Desargues another axiom of the same type (that is, an axiom demanding the automatic realization of the last incidence relation in each realization of a certain configuration which differs from that of Desargues or Pappus). He proves that this is not possible. Each attempt to organize the set of lines of the plane according to the principle that a certain configuration becomes closed leads to a geometry with the condition of Desargues or Pappus. This is shown valid even under more general hypotheses than those given by the ordinary axioms of order and continuity. *L. M. Blumenthal.*

Petkantschin, B. Axiomatischer Aufbau der zweidimensionalen Möbiusschen Geometrie. Ann. Univ. Sofia. Fac. Sci. Livre 1, Math. Phys. **36**, 219–325 (1940). (Bulgarian. German summary) [MF 3357]

Algebraic Geometry

Berzolari, Luigi. Su alcune rappresentazioni del sistema di due cubiche sghembe in posizione ottaedrica. Atti Accad. Sci. Torino **74**, tomo 1, 653–674 (1939). [MF 1731]

Thalberg, Olaf M. Pencils of algebraic curves of genus 1. Avh. Norske Vid. Akad. Oslo **1940**, no. 1, 12 pp. (1940). [MF 3237]

Properties involving the intersections of binodal quartics of a pencil and rational curves, whose singularities occur at all or some of the ten basis points of the pencil, are studied by means of an involution in which the quartics are self-conjugate. Certain results are extended to pencils of elliptic curves of any order. *T. R. Hollcroft* (Aurora, N. Y.).

Pompilj, Giuseppe. Sulle trasformazioni Cremoniane che posseggono per curva di punti uniti una sestica con dieci punti doppi. Bull. Amer. Math. Soc. **46**, 684–686 (1940). [MF 2659]

In an earlier paper [Rend. Sem. Mat. Univ. Roma (4) **2**, 47–87 (1938)] the author presented a proof that a plane Cremona transformation exists, having for curve of invariant points a sextic with ten double points. Coble showed [Bull. Amer. Math. Soc. **45**, 285–288 (1939)] that this transformation is the identity. The present note presents a second proof. If the premises are applicable, this proof is correct. *V. Snyder* (Ithaca, N. Y.).

Huff, Gerald B. A geometry associated with Cremona's equations. Amer. J. Math. **62**, 855–867 (1940). [MF 2885]

If a linear system of plane curves of order x_0 has multiplicities $x_1, \cdots, x_\rho$ at the base points, a Cremona transformation with these base points effects on the vector $(x_0; x_1, \cdots, x_\rho)$, called the characteristic of the system, a linear transformation which leaves invariant the forms $(xx)=x_1^2+\cdots+x_\rho^2-x_0^2$, $(lx)=x_1+\cdots+x_\rho-3x_0$. The author interprets the characteristics as points in projective ρ-space, and considers from the geometric point of view certain types of transformations which leave the two forms invariant. Among the important transformations are the involutions $I_a: x'=x-2a(ax)/(aa)$, where $(la)=0$; in particular the ones for which $(aa)=2$ or -2. These are related to a more complicated type of transformation $S_{a,b}$, where $(aa)=(la)=(lb)=(ab)=0$. It is shown that $I_aI_b=S_{a-b,a}$, and by means of this relation certain results of Barber [Amer. J. Math. **56**, 109–121 (1934)] concerning the permutability of I_aI_b and I_cI_d are greatly simplified. The cases $\rho=9, 10, 11$ are given special treatment, particularly in relation to the determination of characteristics and transformations which are geometrically realizable. For $\rho=10$ a method is given for constructing all P-characteristics ($(xx)=1$, $(lx)=-1$), and it is proved that every linear transformation preserving (xx) and (lx), and having non-negative integer coefficients arises from a Cremona transformation. (This was previously known to be true for $\rho\leqq 9$ and false for $\rho\geqq 11$.)
R. J. Walker (Princeton, N. J.).

Gherardelli, Giuseppe. Sulla linea tacnodale del sistema ∞^1 di coniche di una quadrica che osculano una cubica sghemba. Boll. Un. Mat. Ital. (2) **2**, 320–321 (1940). [MF 2983]

Given a non-conical quadric surface and a pencil of quadric cones with common vertex and a base line on this quadric. The residual pencil of space cubic curves are bitangential and form a quadratic involution, the double elements of which are composite. The two conjugate cubics of each pair have the property that the osculating conics of either one have the other for tacnodal locus. By stereographic projection of the quadric, the configuration discussed by Berzolari [Boll. Un. Mat. Ital. (2) **2**, 1–5 (1939); cf. these Rev. **1**, 300] results. *V. Snyder.*

Piazzolla-Beloch. Sur le nombre des plurisécantes et sur la classification des courbes gauches algébriques. C. R. Acad. Sci. Paris **210**, 655–657 (1940). [MF 3040]

Using the formula derived in 1870 by Zeuthen defining the number of quadrisecants of an algebraic space curve when that number is finite, and the fact that each k-secant is algebraically equivalent to C_4^k quadrisecants, the various algebraic combinations, geometrically possible, of multiple secants for a curve of given order n and genus p are considered types in this method of classification of space curves. Noether employed this classification to obtain subtypes of his intersection types [cf. M. Noether, Abh. Preuss. Akad. Wiss. **1883**, 97, first paragraph]. The author classifies curves only of order seven to exemplify the method. The new feature consists in defining the types by means of the above algebraic relation. The classification is not complete since only curves with a finite number of multiple secants are distinguished by this method. For example, for order seven, two Noether types, $p=0$, [2, 6] and $p=4$, [2, 5], are omitted. *T. R. Hollcroft* (Aurora, N. Y.).

Comessatti, Annibale. Intorno ad un classico problema di unisecanti. Boll. Un. Mat. Ital. (2) **2**, 97–104 (1940). [MF 2965]

An algebraic surface containing a rational pencil of rational curves can be proved rational by establishing the existence of one or more rational curves on the surface unisecant to each of the curves of the pencil. Noether [Math. Ann. **3**, 161–227 (1871)] showed that, aside from the obvious case of a surface with an $(n-1)$-fold line, such a surface can be reduced to the following normal form: a surface F of order n with an $(n-2)$-fold line r and a pencil of conics φ_t cut from F by a pencil of planes through r. Noether obtained on F a definite number of rational curves C, of order $n-2$ and with r as an $(n-3)$-secant, each of which intersects each φ_t in one and only one point P_t. He

then mapped F on a plane by projection, the points of each φ_i being projected by lines through the point P_i common to it and a given C. The methods used by Noether were "elegant, but relatively complicated and not entirely rigorous." In the present paper, by direct algebraic processes, the existence of rational curves on F unisecant to the φ_i is established and their equations obtained.

T. R. Hollcroft (Aurora, N. Y.).

Franchetta, Alfredo. Sulle curve eccezionali riducibili di prima specie. Boll. Un. Mat. Ital. (2) **2**, 332–341 (1940). [MF 2986]

By a well-known result of Castelnuovo-Enriques, the exceptional curves of the first kind on an algebraic surface F are fixed components of the canonical system K. In the present paper a precise formulation of this result is derived in the case of a reducible exceptional curve $\Omega=h_0\omega_0+h_1\omega_1+\cdots+h_s\omega_s$, where the ω_i are the irreducible components of Ω. The curve Ω will correspond to the whole neighborhood of a simple fundamental point O followed by infinitely near fundamental points $O_1, \cdots, O_s$ on some irreducible branch through O. To the whole neighborhood of O_i there will correspond a curve $\Omega^{(i)}$ of the form $h_i^{(i)}\omega_i+\cdots+h_s^{(i)}\omega_s$. The main result is to the effect that the curve $\Omega+\Omega^{(1)}+\cdots+\Omega^{(s)}$ is a fixed component of K, provided O is not a base point of the canonical system. The proof is based on the results of the joint paper by Barber and the reviewer [Amer. J. Math. **57**, 119–141 (1935)]. *O. Zariski.*

Lilley, S. On the isolated united points of a cyclic involution on an algebraic surface. Proc. London Math. Soc. (2) **46**, 312–360 (1940). [MF 2755]

The author treats certain problems associated with the isolated united points O of a cyclic involution I_n on an algebraic surface F. The paper is divided into three parts in which are solved, respectively, the three problems: I. To determine the position and nature of the united points of I_n in the neighborhoods of O. II. Given a linear system $|C|$ of curves on F, satisfying certain conditions, compounded of I_n and not having a base-point at O, to discover the nature of the (virtual) base-point in the neighborhoods of O of subsystems of $|C|$ which have base-points at O. III. To determine the nature of the image point S of a point O on a projective model Φ of the system $|C|$.

Special cases of the above problems have been studied by Godeaux. In the present paper, the solutions are complete. The solution of III, however, depends on the assumption of the actual existence of a certain subsystem of $|C|$ whose virtual existence only was established in II.

T. R. Hollcroft (Aurora, N. Y.).

Emch, Arnold. Zwei Abbildungs-Probleme. Comment. Math. Helv. **12**, 246–253 (1940). [MF 3056]

The first problem is concerned with the plane map of the Steiner surface. Plane sections are mapped by conics, sections by tangent planes by composite conics. For hyperbolic regions of the surface, the components of the image conic are real, for elliptic regions, imaginary. The second problem is the map of a plane quadratic involution on the Cayley cubic surface having four nodes. The plane sections are mapped by a web of cubic curves, each invariant under the involution. Sections by tangent planes are mapped by nodal cubics in the web [Emch, Amer. J. Math. **48**, 21–44 (1916)]. A line of the plane meets the conjugate conic in the involution in a pair of points, images of one point on the surface. When the points are real, the point on the surface is in a hyperbolic region; when imaginary, the point on the surface is still real, and in an elliptic region. *V. Snyder.*

Scott, D. B. Invariant groups associated with an algebraic surface. Proc. Cambridge Philos. Soc. **36**, 414–423 (1940). [MF 2920]

Let $\gamma_1, \cdots, \gamma_{2q}$ be a base for the 1-cycles on an algebraic surface F and let $\Gamma_1, \cdots, \Gamma_{2q}$ be a corresponding base for the 3-cycles, that is, such that the matrix of the intersection numbers $(\gamma_i \cdot \Gamma_j)$ is the unit matrix. If C is any algebraic cycle on F, then $\Gamma_i \cdot C \approx a_{ij}\gamma_j$, where the a_{ij} are integers. Thus with each such C there is associated a matrix $\underline{a}=\|a_{ij}\|$. These matrices form an additive free group which possesses a finite base of at most ρ elements over the set of integers. Here ρ is the number of independent algebraic cycles (it is shown by an example that the number of elements in the base can be actually less than ρ). This group is referred to as the algebraic intersection group of F over the cycles $\gamma_1, \cdots, \gamma_{2q}$. It is shown that this group is not only a topological, but also an absolute birational invariant of F. If C is a sufficiently general curve on F, for instance a plane section of a birational transform of F, then the cycles γ_i can be assumed to be on C, giving rise to intersection numbers $(\gamma_i \cdot \gamma_j)$ on C. It turns out that the matrix $\underline{a}$ is then the inverse of the transposed of the matrix $\|(\gamma_i \cdot \gamma_j)\|$. From this it follows that $\underline{a}$ is a principal matrix of the period matrix of F (relative to the cycles γ_i). *O. Zariski.*

Woodcock, E. R. A class of fundamental $[n-1]$ scrolls. Proc. Cambridge Philos. Soc. **36**, 288–299 (1940). [MF 2530]

The name "scroll" for a ruled surface seems to have been introduced by W. G. Welchmann [cf. in particular Proc. Cambridge Philos. Soc. **29**, 235–244 (1933) and Proc. London Math. Soc. (2) **40**, 143–188 (1936)]. The name "plane scroll" is used for a locus generated by ∞^1 planes, "k-scroll" for a locus of ∞^1 linear manifolds of dimension k. Welchmann studied algebraic scrolls and introduced the idea of "fundamental scrolls," which are generated by joins of pairs of corresponding points in a cyclic (1, 1)-correspondence of period 2 on a canonical curve of genus p. He proved that all special line scrolls may be obtained by projection of fundamental scrolls, the theory of which he developed.

In this paper the author generalizes some of Welchmann's results, and considers certain fundamental scrolls of particular interest. This work was made possible by some recent results of P. Defrise [Acad. Roy. Belgique. Bull. Cl. Sci. (5) **24**, 313–332 (1938)] about cyclic involutions γ_n^1 on algebraic curves. It is known that a (1, 1)-correspondence T on a curve of genus greater than 1 is necessarily cyclic and therefore gives rise to a cyclic involution γ_n^1. The author considers the scrolls formed by spaces joining sets of a cyclic γ_n^1 on a canonical curve $K(p)$ in the case when γ_n^1 is such that all its multiple points are n-ple, and that a generic set of γ_n^1 determines a linear $[n-1]$. These scrolls are called "fundamental" and the theorem is proved: Any special (algebraic) scroll is the projection of a fundamental scroll. The author studies some properties of his fundamental scrolls and gives tables for the possibilities for the lower values of p and for $n=3, 4, 5$. If a canonical curve possesses a birational transformation into itself, it is not the general curve for its genus. The author obtains necessary conditions on the equations of quadrics through the curve for it to possess a self-transformation. These condi-

tions are also sufficient, provided that the curve does not contain a linear g_3^1 or lie on a Veronese surface.

E. Helly (Paterson, N. J.).

Finsler, Paul. Die eindimensionalen Freigebilde. Comment. Math. Helv. **12**, 254–262 (1940). [MF 3057]

A "Freigebilde" is an algebraic manifold with the property that no linear space intersects it in a finite number of linearly dependent points. Irreducible one-dimensional Freigebilde are rational normal curves. Any one-dimensional Freigebilde G consists of such curves and isolated points, and the spaces of lowest dimension containing these curves contain no other point of G. A connected Freigebilde consists of r rational normal curves of degrees $\mu_1, \cdots, \mu_r$ which intersect in at most $r-1$ points and the linear space of lowest dimension containing $c_1+\cdots+c_r$ has dimension $\mu_1+\cdots+\mu_r$. Conversely, any algebraic manifold of this type is a one-dimensional Freigebilde. If the number of connected parts of a Freigebilde G is s and ρ the dimension of the lowest space containing it, then $\rho-s+1$ is the order of G.

H. Busemann (Chicago, Ill.).

Godeaux, Lucien. Sur quelques variétés réglées à trois dimensions. Revista Ci., Lima **42**, 345–354 (1940). [MF 2898]

The three dimensional ruled varieties may be defined as follows. Given two general plane curves, of orders m, n in the same plane. Construct the surfaces of Veronese of the linear systems to which the curves belong, and connect by straight lines those points of the surfaces which are images of the same point in the plane. They generate a three dimensional ruled variety in the minimum space of the two surfaces. The present paper first considers the variety associated with $m=3$, $n=2$ in some detail, and gives an outline of the general case m, n. When $m>n$, the variety is the image of an involution of order $m-n$, under which the system of surfaces of $[3]$, images of the prime sections of the variety, remain invariant.

V. Snyder (Ithaca, N. Y.).

Godeaux, Lucien. Sur quelques variétés algébriques à trois dimensions. Actas Acad. Ci. Lima **2**, 173–178 (1939) = Revista Ci., Lima **42**, 3–8 (1940). [MF 2666]

The author constructs the canonical system of a three dimensional algebraic variety defined as follows. In $[3]$ the totality of algebraic surfaces of order n defines a linear system of dimension $r-1$, which is mapped projectively on the points of $[r-1]$. The planes of $[3]$ become surfaces of order n^2, lying on a variety of Veronese V_3 of order n^3. A surface of order $\nu<n$ becomes a surface of order $n^2\nu$. Let $[r-1]$ be imbedded in $[r]$ containing a point O not on $[r-1]$. Project V_3 and all surfaces from O, and cut by a primal of order m through O. The intersection consists of the projection of the surface and Ω_3, the latter having the canonical system

$$|(mn-n-\nu-4)F+(m-2)\Delta|,$$

in which F is the residual intersection with V_4 of the projection of a plane, and Δ the image of O.

V. Snyder (Ithaca, N. Y.).

MATHEMATICAL PHYSICS

Carathéodory, C. Elementare Theorie des Spiegelteleskops von B. Schmidt. Hamburg Math. Einzelschr. **28**, 36 pp. (1940). [MF 3485]

The author begins with the theory of the Schmidt telescope (spherical mirror+correction plate) for monochromatic light. The equation of the aspheric surface giving a sharp image at the focus of our mirror is given in Cartesian coordinates. For a refractive index of $n=1.5$ there is given a table allowing rays to be traced through the instrument. Introducing a stop of the order of magnitude used in the Schmidt camera, and thus limiting the rays taken into consideration, the author succeeds in giving an important approximate form for the correction plate and the corresponding ray tracing formulae. The second chapter treats the image errors in a Schmidt camera, especially the image of the stop produced by a bundle of parallel rays going through a certain aperture of the stop. The author calculates approximately the curves in which these rays intersect the image plane. Due to the apparent difficulty of the problems, the vignetting effect was mentioned but not taken into consideration. The author investigates further the influence of chromatic aberration.

The importance of the paper, aside from the special problem discussed, rests in the fact that here, for the first time, an attempt has been made to investigate the image of an extra-axial point without the classical division into image errors, and to show that image errors of different order, according to theory, can be of the same order of magnitude in a practical problem.

M. Herzberger.

***Rashevsky, Nicolas. Advances and Applications of Mathematical Biology.** University of Chicago Press, Chicago, Ill., 1940. xiii+214 pp. $2.00.

"Mathematical Biology" is here taken as synonymous with the author's coinage "Mathematical Biophysics," and it denotes the elaboration of quantitative biological theories from premises which describe the properties of and relations among the physical elements, either as observed or as hypothecated. The present work is a description of the research of the author and his associates along these lines.

The first seven chapters which deal with the vegetative and with the reacting (nerve) cell serve to extend certain chapters of classical physics. First a simplified picture of the vegetative cell is set up whereby it is possible to study approximately the diffusion phenomena by means of ordinary differential equations. Hence the consumption rate of, for example, oxygen as a function of the external concentration is deduced and compared with empirical measurements. By postulating diffusion forces and surface tension as leading to cell division (reproduction) a "critical" radius for the cell is deduced beyond which division must occur. The subjects of cell growth, of equilibrium shapes, and of protoplasmic streaming are all considered in greater or less detail. The theory of nerve excitation is purely formal, and no discussion of nerve conduction or transmission is entered into here. The remaining five chapters deal with the nervous system, where postulates of a different order are required. If the dynamics of individual nerve fibers are known (the dependence of the response upon the stimulus) then, conduction and transmission time being neglected, the response of a network of fibers of given topological structure can be predicted, and hence the stimulus-response sequence described for any organism which possesses in its nervous system such a network. Several specific networks are considered, and the predicted stimulus-response sequence is compared with psychological findings.

Mathematical realms as yet unexplored have been re-

peatedly sidestepped by making a simplification in the picture, but it becomes evident from a reading of this book that biology will open up many such realms for the mathematicians as physics has done so often in the past.

A. S. Householder (Chicago, Ill.).

* **Jeans, Sir James. An Introduction to the Kinetic Theory of Gases.** Cambridge University Press, Cambridge, England; Macmillan Company, New York, 1940. vii +311 pp. $3.50.

This book contains a treatment of the whole kinetic theory of gases exclusively from the point of view of classical Newtonian mechanics, the most frequently used model being the swarm of elastic spheres. The physical knowledge presupposed does not go beyond an undergraduate course in general physics. The mathematics does not exceed advanced calculus. The broad scope of the work is shown by the later chapter and appendix headings: IV. Collisions and Maxwell's Law. V. The Free Path in a Gas. VI. Viscosity. VII. Conduction of Heat. VIII. Diffusion. IX. General Theory of a Gas not in a Steady State. X. General Statistical Mechanics and Thermodynamics. XI. Calorimetry. App. I. Maxwell's Proof of the Law of Distribution of Velocities. App. II. The *H*-Theorem. App. III. The Normal Partition of Energy. App. IV. The Law of Distribution of Coordinates. App. V, VI. Tables. On the other hand, no mention is made of Gibbs' canonical ensemble nor of the ergodic theory, some of the conceptions of which "come in by the back door." But no table of contents can convey an impression of the clear and direct manner in which the author expounds the essential physical ideas, illuminating them at every turn by most lucid descriptions of a wealth of crucial experiments bearing directly upon them and drawn as much from traditional methods as from the most strikingly recent technique. The book will supply the mathematician with a rich ground of physical knowledge and conceptualism on the basis of which to undertake a deeper study of advanced statistical mechanics.

B. O. Koopman (New York, N. Y.).

Nagamiya, Takeo. Statistical mechanics of one-dimensional substances. I. Proc. Phys.-Math. Soc. Japan (3) **22**, 705–720 (1940). [MF 2939]

A one-dimensional substance is an assembly of molecules constrained to lie on a one-dimensional variety. This paper studies such substances as simplified models of real three-dimensional ones. Since there can be no question of rigidity, the solid state is distinguished from the liquid by the presence of regularity of molecular arrangement in the former. This makes it possible to envisage three phases in one-dimensional substances. The study is made on the basis of each of three types of assumed molecular interaction, both from the quantum and the non-quantum point of view. The machinery of modern statistical mechanics is brought into play, and interesting analogues of the behavior of bodies are considered. *B. O. Koopman.*

Lichnerowicz, André et Marrot, Raymond. Remarques sur l'équation intégro-différentielle de Boltzmann. C. R. Acad. Sci. Paris **210**, 391–393 (1940). [MF 1986]

Let $F(x, y, z, u, v, w, t)dxdydzdudvdw$ be the number of molecules of a gas which, at time t, have their centers in the volume element $(x, x+dx; y, y+dy; z, z+dz)$, and their velocities in the element $(u, u+du; v, v+dv; w, w+dw)$. F is then a solution of the differential equation of Boltzmann. The authors assume (i) that the force field is derived from a potential, and (ii) that $F=F(x, y, z, r^2, t)$, where $r^2=u^2+v^2+w^2$. They state that F is then of the form $F(\psi, t)$, where $\psi=\frac{1}{2}r^2-U$ (the energy of a molecule of unit mass). In addition, they state (also without proof) conditions under which the Boltzmann equation has one and only one solution $F(\psi, t)$ satisfying $|F|<\text{constant}/(1+r)^\alpha$, $\alpha>6$. *D. C. Spencer* (Cambridge, Mass.).

Tolman, Richard C. On the establishment of grand canonical distributions. Phys. Rev. (2) **57**, 1160–1168 (1940). [MF 2350]

Following Gibbs, by a grand canonical ensemble is meant a system of particles the numbers $n_1, \cdots, n_h$ of whose h possible kinds vary, and with the energy E_i of the system varying, all in accordance with the law of probability

$$P_{n_1,\cdots,n_h,i}=\exp(\Omega+\mu_1 n_1+\cdots+\mu_h n_h-E_i)/\theta,$$

but where fixed known values of such external coördinates as the volume occupied by the system prevail. Here i denotes the quantum state of the system in which the total energy has the characteristic value E_i, and $\Omega, n_1, \cdots, n_h, \theta$ are physical parameters of the system ($\theta=kT$). On the basis of this postulated distribution and of appropriate identifications of statistical with thermodynamical quantities, an "explanation" (that is, isomorphism) can be given for the phenomenological thermodynamic behavior of bodies. But there remains the question as to the possibility of establishing the validity of this law of distribution by a priori arguments.

This paper is a contribution to this question. It establishes the distribution by means of a number of (uncontrolled) mathematical approximations, starting with three assumptions: first, that the Gibbs distribution for canonical ensembles is valid (that is, when the number of component particles is fixed); next, two assumptions concerning the dependence of certain "sum-over-states" quantities on the number of particles. These assumptions are treated in the author's book: The Principles of Statistical Mechanics [Oxford, 1938]. The physical model employed is that of the system proper in contact via semipermeable membranes with h large reservoirs of the different kinds of particles, as well as a large heat reservoir. *B. O. Koopman.*

van Dantzig, D. On the thermo-hydrodynamics of perfectly perfect fluids. I. Nederl. Akad. Wetensch., Proc. **43**, 387–402 (1940). [MF 1915]

The fluid under consideration is a mixture of n components in relative equilibrium with one another, so that they all have the same 4-velocity i^h; an electromagnetic field may be present. The theory is set up in a curved metrical space-time, but the later developments are non-metrical in the sense that they do not involve the metric tensor explicitly. The total tensor-density of stress, momentum and energy is taken to be

$$\mathfrak{T}^h_{\cdot i}=\mathfrak{P}^h_{\cdot i}-\mathfrak{s}^h\phi_i+\mathfrak{S}^h_{\cdot i},$$

where the terms in order arise from the fluid alone, the interaction of the fluid and the field, and the field alone; $\mathfrak{S}^h_{\cdot i}$ is the usual electromagnetic energy tensor-density, $\mathfrak{s}^h$ the electric current-density, ϕ_i the 4-potential and $\mathfrak{P}^h_{\cdot i}$ the energy tensor-density of a perfectly perfect fluid [cf. van Dantzig, these Rev. 1, 96]; it has the form

$$\mathfrak{P}^h_{\cdot i}=\mathfrak{D}^h\mathfrak{p}_i+\mathfrak{p}\delta^h_{\cdot i},$$

where $\mathfrak{p}$ is the pressure-density and $\mathfrak{D}^h$ the temperature vector, $\mathfrak{D}^h=ci^h/kT_0$, where k is Boltzmann's constant and

T_0 the proper temperature; $\mathfrak{p}$ is regarded as a function of $n+8$ variables, namely, the coordinates x^h, the temperature components $\mathfrak{D}^h$ and n quantities λ^r, which are chemical potentials. (The suffix r takes the n values $5, \cdots, n+4$.) The quantities $\mathfrak{p}_r$, $\mathfrak{p}_i$ are defined by $\mathfrak{p}_r=\partial\mathfrak{p}/\partial\lambda^r$, $\mathfrak{p}_i=\partial\mathfrak{p}/\partial\mathfrak{D}^i$, and explicitly

$$\mathfrak{p}_i=-c^{-1}kT_0(\rho+\mathfrak{p})i_i+\mathfrak{p}_r e^r\phi_i,$$

where ρ is the material density and e^r the charge on each of the particles of the rth component of the fluid.

Assuming that $\mathfrak{p}(-g)^{-\frac{1}{2}}$ is a function only of the $n+1$ quantities $\mathfrak{D}_0=c/kT_0$, $\lambda^r+e^r\phi_i\mathfrak{D}^i$, the author obtains, from the vanishing of the divergence of $\mathfrak{T}^h_{.i}$ and Maxwell's equations, the non-metrical equations of motion

$$\partial_j\mathfrak{P}^j_{.i}-\partial_i{}^0\mathfrak{p}=0,$$

where $\partial_j=\partial/\partial x^j$ and $\partial_j{}^0$ the same operator under constant $\mathfrak{D}^h$ and λ^r. As an alternative form of these equations the author gives

$$\frac{d_L\mathfrak{p}_i}{d\theta}=-\mathfrak{p}_r\partial_i\lambda^r,$$

where d_L indicates a Lie derivative with respect to the vector field $\mathfrak{D}^h$.

The rest of the paper is devoted to variational principles, of which the principal one is

$$\delta_L{}^0\int_U\mathfrak{p}d\Omega=\int_B\mathfrak{P}^h_{.i}\delta x^i d\mathfrak{B}_h,$$

where in the Lie variation $\delta_L{}^0$ the variations $\delta_L{}^0\mathfrak{D}^h$, $\delta_L{}^0\lambda^r$ are zero; the integral on the left extends over a portion U of space-time and that on the right over its boundary B. Among the other variational principles there is included (as a particular metrical case) that of Eisenhart [Trans. Amer. Math. Soc. 26, 214–216 (1924)]. *J. L. Synge.*

van Dantzig, D. On the thermo-hydrodynamics of perfectly perfect fluids. II. Nederl. Akad. Wetensch., Proc. 43, 609–618 (1940). [MF 3103]

Continuing an earlier paper [see preceding review], the author deduces from his equations of motion that for a perfectly perfect fluid, composed of several chemical components, an increase of entropy is possible only if the amount of at least one component changes. The entropy is measured by the flux of an entropy current $\mathfrak{S}^h=-\mathfrak{D}^h(\mathfrak{p}_i\mathfrak{D}^i+\mathfrak{p}_r\lambda^r)$ across an element of volume "dragged along" the macroscopic world-lines of flow. He discusses mass-action in connection with isentropy. In the absence of chemical reactions, his equations of continuity merely express the numerical conservation of the particles of the several components. Equations of continuity and definitions of incompressibility used by various writers are criticized. These the author regards as erroneous, and he remarks (a) that many of them are consequences of the habit of treating thermodynamics and hydrodynamics as separate disciplines, (b) that many of them concern only small, though typically relativistic, corrections, (c) that in classical hydrodynamics they are used sometimes consciously as approximations only, (d) that his method of treatment (invariant and independent of metrical geometry) makes it impossible to commit most of them unconsciously. In the last part of the paper a formal simplicity is given to the equations of motion and continuity by the use of a space of $n+4$ dimensions (n is the number of chemical components present); a still more symmetric formalism is obtained by using $n+5$ homogeneous coordinates in this space. *J. L. Synge* (Toronto, Ont.).

Huxley, L. G. H. The lateral diffusion of a stream of ions in a gas. Philos. Mag. (7) 30, 396–413 (1940). [MF 3691]

The theoretical problem of determining the space distribution of ions or electrons in their steady state of motion through a gas in a uniform electric field is solved by finding the appropriate solution of the equation

$$\frac{\partial^2 n}{\partial x^2}+\frac{\partial^2 n}{\partial y^2}+\frac{\partial^2 n}{\partial z^2}=2\lambda\frac{\partial n}{\partial z}.$$

Here n is the number of ions or electrons per cubic centimeter and λ is a constant, and the z-axis is taken in the direction of the field. Formulas are derived for the distribution at one of two parallel plane electrodes perpendicular to the z-axis when the ions enter through a small circular hole, or through a slit, in the other electrode. The method of images is used to obtain the desired solution of the above equation. In the last section of the paper the effect of magnetic fields is examined. *R. V. Churchill.*

Schuchard, E. A. and Uehling, E. A. Steady-state diffusion under conditions of generalized source and incident current distributions. Phys. Rev. (2) 58, 611–623 (1940). [MF 2857]

A problem in the diffusion of neutrons in a semi-infinite medium is attacked with the aid of the theory of functions of a complex variable. The intention is to derive a formula for the current at the boundary and to discuss the consequences of this formula. The plan and details of the mathematical development are not made entirely clear.

R. V. Churchill (Ann Arbor, Mich.).

Kramers, H. A. Brownian motion in a field of force and the diffusion model of chemical reactions. Physica 7, 284–304 (1940). [MF 1928]

This paper is in the classical Einstein vein: that is, it treats the Brownian motion as a random sequence of impulses on a particle, unaffected by the motion of the particle itself, and characterizes the randomness of these impulses by a sequence of moments. The dynamical cause of the distribution of impulses is not studied. The differential equations of a system in which such random disturbances are superimposed on given laws of force are studied, both when the fixed forces predominate and when the random impulses predominate, and the results are used to discuss the manner in which a particle in a trough of potential may be pushed over the edge of the trough by random impulses.

N. Wiener (Cambridge, Mass.).

Balandin, A. A. Structural algebra in chemistry. Acta Physicochim. URSS 12, 447–479 (1940). [MF 2665]

This paper is concerned with the representation of chemical and physical data related to atomic and molecular systems in the form of matrices. In some cases the results display the mathematical properties of matrices.

J. W. Tukey (Princeton, N. J.).

Balandin, A. A. Calculation of the intermediate equilibria in catalytic kinetics and the geometry situs. C. R. (Doklady) Acad. Sci. URSS (N.S.) 24, 741–747 (1939). [MF 2060]

If a number of substances are in equilibrium through unimolecular reactions which form a linear graph, then the concentration of any one can be written as a quotient in terms of the total concentration and the rate constants of

the various reactions in such a way that the structure of the numerator is associated with the same linear graph, while the structure of the denominator is simple. This useful process can be extended to some bimolecular reactions.

J. W. Tukey (Princeton, N. J.).

Pipes, Louis A. The matrix theory of four-terminal networks. Philos. Mag. (7) **30**, 370–395 (1940). [MF 3690]

Epstein, D. W. and Donley, H. L. The application of the tensor concept to the complete analysis of lumped, active, linear networks. RCA Rev. **4**, 73–82, 240–252 (1939). [MF 3535]

A generalization of the Carson integral equation is presented, with the aid of tensor notation, which applies to any multi-mesh linear electrical network of lumped elements, with any initial conditions and with any applied electromotive forces in the meshes. The process of solution is similar to that usually employed in the operational calculus, using Laplace transforms. The method is illustrated by several numerical examples. *R. M. Foster.*

Neville, E. H. The mutual inductance of two circles: a problem in approximation. Philos. Mag. (7) **30**, 340 (1940). [MF 3185]

The approximation

$$M=4\pi ab\int_0^\pi \frac{\cos\theta\,d\theta}{r}=4\pi c\{\log(8c/q)-2\}+O(q)$$

is derived for the mutual inductance M of two coplanar concentric circles of radii a, b, where $r^2=a^2+b^2-2ab\cos\theta$, $ab=c^2$, $b-a=q$. This approximation has some advantage over a similar formula given by Maxwell [Electricity and Magnetism **2**, 342]. *H. Poritsky* (Schenectady, N. Y.).

Spiwak, G. V. and Zrebny, P. E. Influence of magnetic field upon electron motion in axially symmetrical fields. C. R. (Doklady) Acad. Sci. URSS (N.S.) **24**, 242–246 (1939). [MF 2911]

This paper generalizes previous work by the authors [Tech. Phys. USSR **9**, 715 (1938); C. R. (Doklady) Acad. Sci. URSS (N.S.) **24**, 237 (1939)] on the electron current between coaxial cylindrical electrodes when a homogeneous magnetic field is applied along the axis of the cylinders. The present paper removes the restriction that the radius of the outer cylinder should be large compared with that of the inner, and formulas are derived for the currents when the electrons are emitted either from the outer or from the inner cylinder. It is shown that the two cases are distinct, the currents in the latter case being always larger than in the former. *M. C. Gray* (New York, N. Y.).

Chapman, S. and Ferraro, V. C. A. The theory of the first phase of a geomagnetic storm. Terr. Magnetism **45**, 245–268 (1940). [MF 2764]

The authors consider an idealized cylindrical-shell problem in which, with cylindrical coordinates r, ϕ, z, the idealized permanent magnetic field has only a z-component which is a function of r alone and is indeed, for $r>a$, equal to $H_0(a/r)^n$, where $n>2$ and $a=6.37\times10^8$ cm., the earth's radius, while $H_0a^3=8\times10^{25}$ (the earth's magnetic moment). The system of particles which moves in this permanent field is supposed to be distributed initially over the cylinder $r=r_0$ in a uniform way along each generator, a particle having the initial velocity components $(-u_0, 0, 0)$. The velocity u_0 is taken to be 10^8 cm. per second and r_0 is taken to be the distance between the earth and the sun (1.5×10^{13} cm.). The particles are supposed to consist of electrons of mass m_e and positive ions of mass m_i. The amount of charge, of either sign, per unit-length of the cylinder $r=r_0$ is taken to be $\pm Q$. An unseparated sheet of charges lags behind an uncharged sheet moving with constant velocity u_0 because of the retardation of the ionized sheet. The tendency for u_i to become different from u_e comes from the unequal action of the magnetic field on the ions and electrons once they have acquired transverse (azimuthal) motions by the deflecting action of the field. Each particle moves in a plane normal to the z-axis and there is a type of plane orbit for each type of particle. Solutions of the field equations are obtained for the varying field. Equations of motion are written down for the ions and electrons and a discussion is given of the motion before and after separation. The solution depends on an incomplete gamma function of type

$$I_{n+1}(y)=s2^{1-2s}\int_0^u t^{-s}(1-t)^{-s-1}dt, \qquad 2ns=1,$$

where $2u=1-(1-y^{2n})^{\frac{1}{2}}$ and r_0 has been assumed infinite. The whole time lag up to the distance ρ at which the radial motion is reduced to zero depends on $I_s(1)$. The distance at which separation of the two charged sheets occurs is denoted by r_s and in the numerical discussion the condition of no separation $\rho>r_s$ is found to be $Q>(n-1)m_iu_0^2/e$. Values are found also for the minimum distance ρ, the maximum magnetic disturbance and the duration of the magnetic disturbance. The problem is finally discussed in relation to actual magnetic storms. *H. Bateman.*

Graffi, Dario. Sopra le condizioni di Love per un'onda elettromagnetica. Rend. Sem. Mat. Univ. Padova **10**, 81–89 (1939). [MF 1738]

Verfasser betrachtet die Fortpflanzung eines elektromagnetischen Feldes, das eine endliche Anzahl von Unstetigkeitsflächen aufweist und gewinnt durch eine elementare Differentialbetrachtung des Feldes Gleichungen für die Normalkomponenten der Vektoren für die dielektrische Verschiebung, für die elektrische Feldstärke, für die magnetische Induktion und für die magnetische Feldstärke. Durch Anwendung der Maxwellschen Differentialgleichungen gewinnt er in analoger Weise Gleichungen für die Tangentialkomponenten der genannten Vektoren. Er zeigt, dass die Fortpflanzung der Unstetigkeitsfläche mit Lichtgeschwindigkeit stattfinden muss. Endlich wendet er sich einer Erweiterung des Reziprozitätssatzes für elektromagnetische Felder zu. Diese Erweiterung bezieht sich, wie die obigen Ausführungen, auf die Fortpflanzung von Unstetigkeitsflächen des Feldes. *M. J. O. Strutt* (Eindhoven).

Westerdijk, T. Über inhomogene ebene Wellen. Ann. Physik **36**, 696–736 (1939). [MF 1302]

The author starts with a discussion of the propagation and the reflection including absorption of homogeneous electromagnetic waves on the basis of Maxwell's equations. Nonhomogeneous waves are considered and the rules of their propagation and reflection in transparent media are derived. Then the properties of such waves in absorbing media are discussed. The difference from the usual course of discussion lies in the fact that the author splits the primary waves into real and imaginary waves having different directions of propagation. The three fundamental equations of metal optics are derived in a new way by transformation

into homogeneous non-absorbed oscillations fulfilling a certain set of conditions. The expressions for Poynting's vector are derived and by geometric interpretations give rise to elaborate considerations on the directions of oscillation of the electric and magnetic field strengths. The author then discusses the direction of light rays and the direction of Poynting's vector. A complete account of the energy balance is given. The definitions of the optical indices of refraction and of propagation are discussed. The relations of the author's equations to the refraction laws of Fresnel and Snellius are set forth, leading up to a discussion of refraction under critical angles. The case of total reflection is considered and the directions and absorption coefficients of the different rays discussed. *M. J. O. Strutt* (Eindhoven).

Markov, M. Some contributions to the quantum field theory. C. R. (Doklady) Acad. Sci. URSS (N.S.) **24**, 233–236 (1939). [MF 2910]

A new commutation relation is introduced into the formalism of quantum electrodynamics between the phase ϕ_s of the particle wave with wave number k_s and the position-time coordinate x_μ ($\mu=1, 2, 3, 4$) of the form

$$\phi_s x_\mu - x_\mu \phi_s = 2\pi r_{s\mu}.$$

If $r_{s\mu}$ is put equal to $b^2 k_s$, where b is the square of a universal length r_0, some of the divergent expressions in quantum electrodynamics, for example, the self energy of the electron, become finite. The new relation introduces a "cutting off" factor of the form $\exp(-bk_s^2)$ into the integrals over the momentum space. *V. Weisskopf* (Rochester, N. Y.).

McCrea, W. H. Quaternion analogy of wave-tensor calculus. Philos. Mag. (7) **30**, 261–281 (1940). [MF 3182]

The author gives the details of an analogy to Eddington's wave-tensor calculus by replacing the E-number algebra by the quaternion algebra. The results of this paper are included in the work of Weyl and Brauer [Amer. J. Math. **57**, 425–449 (1935)]. The former algebra is the algebra underlying the four component spin representation of orthogonal groups in a Euclidean space of 4 and 5 dimensions. The latter algebra is that underlying the two component spin representation of orthogonal groups in 2 and 3 dimensions. *A. H. Taub* (Princeton, N. J.).

Datzeff, A. Interprétation quantique de certains résultats en mécanique classique. Ann. Univ. Sofia. Fac. Sci. Livre 1, Math. Phys. **36**, 201–218 (1940). (Bulgarian. French summary) [MF 3355]

Fuchs, K. Operator calculus in the electron theory of metals. Proc. Roy. Soc. London. Ser. A. **176**, 214–228 (1940). [MF 3361]

In the treatment of the wave function for electrons in metals by the ionic polyhedron method the coordinate vector x is not represented by a regular operator since neither its components nor the admitted wave functions vanish at the boundaries. It is shown that x can be replaced by another operator, corresponding to its Fourier development plus certain surface operators having the character of δ-functions at the boundaries. A consistent perturbation calculus can be developed with the help of these quantities. It is applied to the question of the Fermi energy in metals and the results of Wigner and Seitz [Phys. Rev. **43**, 804 (1933)] are verified by the new method. *L. W. Nordheim* (Durham, N. C.).

Born, Max and Fuchs, Klaus. Reciprocity. Part II: Scalar wave functions. Proc. Roy. Soc. Edinburgh **60**, 100–116 (1940). [MF 2181]

A Fourier integral is formed over the relativistically invariant hypersurfaces $r^2-t^2=R^2$, $E^2-p^2=P^2$ and the relativistic scalar products $(\psi_1, \psi_2)_R$, $(\phi_1, \phi_2)_P$ are introduced as integrals over these two hypersurfaces, respectively. For normalized plane waves the scalar products are in general non-analytic functions and a distinction is drawn between region I for which $r^2>t^2$ and region II for which $r^2<t^2$. The Fourier coefficient is defined on R by the scalar product for I and by its analytical continuation for II. The wave equation satisfied by $\psi(r, t)$ is

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}-\frac{\partial^2}{\partial t^2}-\frac{\epsilon^2}{\hbar^2}\right)\psi=0.$$

A transformation to four-dimensional polar coordinates is discussed and pseudo-harmonics are constructed for 4-space. If $x=r\sin\beta\cos\gamma$, $y=r\sin\beta\sin\gamma$, $z=r\cos\beta$, $t=R\operatorname{ch}\alpha$, $t=R\operatorname{sh}\alpha$, solutions of (1) of type

$$R\psi_{klm}=Z_{k+1}(i\epsilon R/\hbar)\Pi_k^l(\tanh\alpha)P_l^m(\cos\beta)e^{im\gamma},$$

and corresponding functions ϕ_{klm} are considered. The wave operator

$$P^2=\hbar^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}-\frac{\partial^2}{\partial t^2}\right)$$

is expressed in terms of the angular momentum and a corresponding expression is found for the wave operator in p-space

$$R^2=-\hbar^2\left(\frac{\partial^2}{\partial p_x^2}+\frac{\partial^2}{\partial p_y^2}+\frac{\partial^2}{\partial p_z^2}-\frac{\partial^2}{\partial E^2}\right).$$

In order to obtain the Fourier transformation of the proper functions it was necessary to obtain evaluations of the integrals

$$\Phi_{kl}=\int_{-\infty}^{\infty}e^{-u}F_l(v)\Pi_k^l(\tanh\alpha)\operatorname{ch}^2\alpha d\alpha,\quad u=ip_0x^0/\hbar;\ v=pr/\hbar,$$

$$\Psi_{kl}=\int_0^{\infty}F_l(v)\Pi_k^l(\coth\alpha')[e^u+(-1)^{k+l+1}e^{-u}]\operatorname{sh}^2\alpha' d\alpha',$$

$$F_l(v)=J_{l+\frac{1}{2}}(v)\cdot v^{-\frac{1}{2}}.$$

In this work aid was obtained from E. T. Whittaker and E. T. Copson. *H. Bateman* (Pasadena, Calif.).

Born, Max and Fuchs, Klaus. Reciprocity. Part III: Reciprocal wave functions. Proc. Roy. Soc. Edinburgh **60**, 141–146 (1940). [MF 2364]

Imaginary values of R as well as of P are now considered and the scalar products defined in Part II [cf. the preceding review] for real values of these quantities are extended by analytical continuation. The wave-equations

$$P^2\psi=(p^2-E^2)\psi=b^2\psi,\quad R^2\phi=(r^2-t^2)\phi=a^2\phi$$

are now considered, where a and b are either real or imaginary. Solutions are obtained as products of a Bessel function and a pseudospherical harmonic in four dimensions. A search for reciprocal functions leads to a discussion of the equation $2\pi\mu J_{k+\frac{1}{2}}^2(\mu)=1$, in which a, b and hence μ are either real or imaginary. A table of real solutions $\mu_{k,n}$ is given for $n=0(1)11$ and $k=-1(1)7$, the number of decimal places being generally 3. The asymptotic solutions for $n\gg k$ are

$$\mu_{k,2n+1}=\tfrac{1}{4}\pi(2k+3+4n-\tfrac{1}{2}),\quad \mu_{k,2n+2}=\tfrac{1}{4}\pi(2k+3+4n+\tfrac{1}{2}).$$

The reciprocal functions found are finally normalized. Other reciprocal functions are formed by linear combinations of those already found. Each term in such an expansion is supposed to represent a kind of explosive process at the origin (an arbitrary point in space-time) and the interpretation may follow lines similar to those in the ordinary theory of collisions, where the incident wave is developed into a series of spherical waves. *H. Bateman.*

Fuchs, Klaus. Reciprocity. Part IV: Spinor wave functions. Proc. Roy. Soc. Edinburgh **60**, 147–163 (1940). [MF 2365]

Commutation relations are given for the angular momentum operator $M^{kl}=x^kp^l-p^lx^k$, where $x^k=(t, x, y, z)$, $p_k=(-E, p_x, p_y, p_z)$ are operators satisfying the commutation relations

$$[p^k, x^l]=\frac{\hbar}{i}g^{kl}, \quad [p_k, x^l]=\frac{\hbar}{i}g_k{}^l,$$

and g^{kl} is the metric tensor. With the aid of these relations it is possible to find the effect of M^{kl} on the proper solutions of the Klein-Gordon equations found in Part II. If

$$x^1=R\,\mathrm{ch}\,\alpha\sin\beta\cos\gamma, \quad x^2=R\,\mathrm{ch}\,\alpha\sin\beta\sin\gamma,$$
$$x^3=R\,\mathrm{ch}\,\alpha\cos\beta, \quad x^0=R\,\mathrm{sh}\,\alpha,$$
$$p_1=P\,\mathrm{sh}\,\alpha'\sin\beta'\cos\gamma', \quad p_2=P\,\mathrm{sh}\,\alpha'\sin\beta'\sin\gamma',$$
$$p_3=P\,\mathrm{sh}\,\alpha'\cos\beta', \quad p_0=P\,\mathrm{ch}\,\alpha',$$

the parts of the proper solutions depending on angles are $f_{k,l,m}(\alpha, \beta, \gamma)$ and $f'_{k,l,m}(\alpha', \beta', \gamma')$, respectively, and the effects of the components of M on these factors are expressed as simple linear combinations of similar factors with the same suffix k and changes of at most one unit in l and m. Similar expressions are obtained for the effect of multiplying by x, y and z, only the suffix k is now changed by at most one unit. The coefficients in each case depend on k, l, m.

The Dirac equation is taken in Cartan's form

$$\{g^kp_k-g\epsilon\}\psi=0,$$

where the g^k are matrices satisfying the commutation relations

$$\frac{g^kg^l+g^lg^k}{2}=g^{kl}, \quad gg^k+g^kg=0, \quad g^2=1.$$

The iteration of this form of Dirac equation leads to the Klein-Gordon equation, while the iteration of $\{g_kx^k-iag\}\phi=0$ leads to the wave-equation $(r^2-t^2)\phi=a^2\phi$. The spin, which is added to the mechanical angular momentum to obtain entities which will compute with the Hamiltonian, forms 3 components of a 6-vector which may be defined as

$$s^{kl}=\frac{g^kg^l-g^lg^k}{2i},$$

and which may be expressed also with the aid of a representation of Pauli's spin matrices. Just as M^{kl} was split up into two 3-vectors M and N, s^{kl} is split up into s^M and s^N and operators ${}^*M=M+\frac{1}{2}\hbar s^M$, ${}^*N=N+\frac{1}{2}\hbar s^N$ are formed as well as their squares ${}^*M^2$, ${}^*N^2$ and the invariant ${}^*K^2={}^*M^2-{}^*N^2$. Commutation relations are found and it is shown in particular that ${}^*M_x$, ${}^*M^2$ and ${}^*K^2$ commute with each other and can be brought simultaneously into diagonal form. The simultaneous proper functions and corresponding proper values are found, the latter being expressed in terms of three half-integers. With their aid a classification is given for the solutions of Dirac's equation which are quadratically integrable on the hyperboloid $r^2-t^2=a^2$. *H. Bateman.*

Rosenfeld, Léon. Sur le tenseur d'impulsion-énergie. Acad. Roy. Belgique. Cl. Sci. Mém. Coll. in 8°. **18**, fasc. 6, 30 pp. (1940). [MF 2964]

In der allgemeinen Relativitätstheorie werden die Feldgleichungen aus einem Variationsprinzip

$$\delta\int(\mathfrak{G}+\mathfrak{L})dw=0, \qquad dw=dx^1dx^2dx^3dx^4,$$

abgeleitet, worin $\mathfrak{G}$ aus der Dichte des Krümmungsskalars durch eine partielle Integration gewonnen ist, während die skalare Dichte $\mathfrak{L}$ neben den g_{lm} noch gewisse materielle Feldgrössen Q_γ und ihre ersten Ableitungen enthält. Sind unter den Q_γ auch Spinoren enthalten, so führt man explizite die Vierbeingrössen $h_j^{(\mu)}$ ($\mu=1, 2, 3, 4$) ein, um $\mathfrak{L}$ allgemein kovariant zu machen. Der Energie-Impulstensor ist dann gegeben durch

$$\tau^{lm}\equiv\tau^{ml}\equiv-\left(\frac{\delta\mathfrak{L}}{\delta g_{lm}}+\frac{\delta\mathfrak{L}}{\delta g_{ml}}\right),$$

bzw., wenn unter den Q_γ Spinorgrössen enthalten sind, durch

$$\overset{j}{\tau}_{\cdot i}=-\frac{\delta\mathfrak{L}}{\delta h_j^{(\mu)}}h_i^{(\mu)}.$$

In letzterem Fall ist die Symmetrie des Energie-Impulstensors zwar nicht identisch erfüllt, wohl aber als Folge der Feldgleichungen. Diese haben auch stets die Kontinuitätsgleichung $\tau^i_{i,j}=0$ zur Folge. Es ist das Ziel der vorliegenden Arbeit, diesen Energie-Impulstensor für allgemeines $\mathfrak{L}$ so umzuformen, dass der resultierende Ausdruck im Falle der speziellen Relativitätstheorie keine Bezugnahme auf die g_{lm} und die $h_i^{(\mu)}$ enthält. Zu diesem Zweck betrachte man die Variation der Grössen Q_ω bei einer infinitesimalen Koordinatentransformation $\delta x^i=\xi^i(x)$ und setze

$$\delta Q_\omega=\overset{j}{c}_{\omega,i}\frac{\partial\xi^i}{\partial x^j},$$

$$d_\omega^{ij}=-d_\omega^{ji}=\tfrac{1}{2}(\overset{i}{c}_{\omega,k}g^{kj}-\overset{j}{c}_{\omega,k}g^{ki}).$$

Ferner mögen bei einer die g_{lm} festlassenden Transformation der Beingrössen

$$\delta h_i^{(\mu)}=d_{i(\lambda\nu)}^{(\mu)}\epsilon^{(\lambda\nu)} \qquad \text{mit } \epsilon^{(\lambda\nu)}=-\epsilon^{(\nu\lambda)}$$

die Q_σ sich gemäss $\delta Q_\sigma=d_{\sigma(\lambda\nu)}\epsilon^{(\lambda\nu)}$ transformieren, woraus die Grössen

$$d_\sigma^{(k)ik}=-h^{(\lambda)i}h^{(\nu)k}d_{\sigma(\lambda\nu)}$$

gebildet werden. Man setze nun $s_\tau{}^{ji}=d_\tau{}^{ji}$ für Tensorgrössen Q_τ, und $s_\sigma{}^{ji}=d_\sigma{}^{ji}+d_\sigma{}^{(k)ji}$ für Spinorgrössen Q_σ. Die $\overset{j}{c}_{\omega,i}$, $d_\omega{}^{ij}$ und $s_\sigma{}^{ij}$ hängen stets linear von den Q ab. (Im Falle der Lorentztransformationen $\delta x^i=\epsilon_k{}^ix^k$ mit $\epsilon_{ik}=-\epsilon_{ki}$ hat man einfach $\delta Q_\alpha=s_\alpha{}^{ij}\epsilon_{ji}$.) Dann wird bewiesen

$$\text{(I)} \qquad \overset{j}{\tau}_{\cdot i}=\frac{\partial\mathfrak{L}}{\partial Q_{\alpha||j}}Q_{\alpha||i}-\mathfrak{L}\delta_i^j-\mathfrak{R}_{i||k}^{kj},$$

$$\text{(II)} \qquad \mathfrak{R}_l^{kj}g^{li}=\mathfrak{B}^{j;ik}-\mathfrak{B}^{i;kj}+\mathfrak{B}^{k;ji},$$

$$\text{(III)} \qquad \mathfrak{B}^{k;ij}\equiv\tfrac{1}{2}(\mathfrak{R}_l^{ki}g^{lj}-\mathfrak{R}_l^{kj}g^{li})=\frac{\partial\mathfrak{L}}{\partial Q_{\alpha||k}}s_\alpha{}^{ij}.$$

Als Anwendung auf den Sonderfall der speziellen Relativitätstheorie werden einfache Folgerungen für die Werte der Volumintegrale von Energie, Impuls und Impulsmoment sowie als Beispiele die Dirac'schen (Spin 1/2) und Proca'schen (Spin 1) Gleichungen diskutiert. Diese Anwendungen sind in Übereinstimmung mit den Unabhängig

von ihnen gewonnenen Resultaten von F. Belinfante [Physica 6, 887–898 (1939); diese Rev. 1, 280].

W. Pauli (Princeton, N. J.).

Pauli, W. Über die Invarianz der Dirac'schen Wellengleichungen gegenüber Ähnlichkeitstransformationen des Linienelementes im Fall verschwindender Ruhmasse. Helvetica Phys. Acta 13, 204–208 (1940). [MF 2651]

In this paper it is shown that the Dirac equations when written so as to be invariant under the transformations of general relativity are invariant under the general conformal transformation: $g'_{ij}=f^2(x)g_{ij}$, provided that the mass of the particle of spin one-half vanishes. The result is established by computing the spin affine connection in the new frame of reference. This method differs from the method used by the reviewer in proving the same result for inversions [Bull. Amer. Math. Soc. 44, 860–865 (1938)]. *A. H. Taub.*

Kakinuma, U. Foundation of the theory of elementary particles. Tensor 3, 25–29 (1940). (Japanese) [MF 2200]

In this paper the author investigates the internal structure of an electron using modifications of the expression of the Lorentz force. He takes the origin of the coordinate system at the center of an electron and considers the ideal case in which the influence of an external electro-magnetic field may be neglected. In this case it is natural to assume that the electron is a three-dimensional closed sphere in some four-dimensional space $\overset{*}{V}_4$. For the natural gouge of Eddington the author puts $\overset{*}{G}=\alpha$, considering the scalar curvature of $\overset{*}{V}_4$ as a constant. In the expression of a variation of length

$$dl=-l\phi_\rho dx^\rho$$

due to Weyl, the author puts in the inner part of the electron

$$\phi_\rho=\frac{\partial \log (E/\epsilon)^{\frac{1}{4}}}{\partial x^\rho},$$

where E denotes the invariant energy-density and ϵ is the constant value of E in $\overset{*}{V}_4$. From this principle the energy tensor in the inner part of the electron is calculated and a physical interpretation of the wave function in wave mechanics is given. *A. Kawaguchi* (Sapporo).

de Wet, J. S. On the spinor equations for particles with arbitrary spin and rest mass zero. Phys. Rev. (2) 58, 236–242 (1940). [MF 2614]

The Fierz-Pauli theory of particles of higher spin is reexamined in the case when no forces are present, and in particular for the rest-mass 0. By a consequent use of the spinor calculus some results of Fierz are obtained in a considerably simpler way: the existence of only 2 "really" independent plane waves (for positive rest-mass their number would be $2j+1$, if the spin is j), and the fact that these waves correspond to spin components $\pm j$ along the momentum vector of the plane wave. *H. P. Robertson.*

Born, M. and Fürth, R. The stability of crystal lattices. III. An attempt to calculate the tensile strength of a cubic lattice by purely static considerations. Proc. Cambridge Philos. Soc. 36, 454–465 (1940). [MF 3051]

The stability of a crystal lattice, strongly stressed and deformed in one particular direction, against any small deformation in other directions, is investigated. General stability criteria are given and explicit calculations are made for a face centered cubic lattice with power law attractive and repulsive forces between the atoms. The breaking strength derived from the above condition comes out to be of the order of the heat of sublimation per unit volume similar to that in the older theories of Polanyi and Zwicky. This is far too high, and it is suggested that the heat motion of the crystal should be taken into account. Then the density of the free energy should be introduced in place of the normal energy density and it is expected that such a theory would give a breaking strength of the order of magnitude of the heat of fusion in accord with the experimental facts.

L. W. Nordheim (Durham, N. C.).

Born, Max and Misra, Rama Dhar. On the stability of crystal lattices. IV. Proc. Cambridge Philos. Soc. 36, 466–478 (1940). [MF 3052]

Methods are developed for estimating the energy density of a highly strained crystal lattice including higher than second order terms in the deformations. Tables for auxiliary functions are given which make possible the computation of all lattice sums for cubic lattices of the Bravais type. Detailed results are given for the third order energy terms for laws of force which can be expressed as sums of terms proportional to powers of the reciprocal distances between pairs of atoms. *L. W. Nordheim* (Durham, N. C.).

Fock, V. A. and Kolpinsky, V. A. Diffraction of waves from a curved lattice. Acad. Sci. U.S.S.R. J. Phys. 3, 125–140 (1940). [MF 3729]

From Laue's theory of X-ray diffraction it is known that, for a three-dimensional crystal-lattice, there are three conditions on the 3 direction cosines of the diffracted ray, which with monochromatic radiation cannot all be fulfilled. With polychromatic radiation, the necessary latitude for the fulfillment of the Laue conditions may be procured by one or the other wave length. Now in diffracting electrons through thin mica crystals Kikuchi discovered a certain diffraction pattern similar to that of a two-dimensional lattice ("crossed gratings"). Yet the crystals contained quite a number of parallel net planes, and as the electron rays were monochromatic (in de Broglie's sense) Kikuchi's result is at variance with Laue's theory. Two explanations have been offered by W. L. Bragg. One explanation considers the mica sheet as a bent lattice, and assumes a concurrent action of different kinds of net planes in the crystal in forming one spot. The authors examine this theory but do not consider a three-dimensional lattice. Consequently they obtain a diffraction pattern of parallel streaks. A flat lattice would produce sharp points instead of the streaks. The difference is due to the different behavior of the maxima of intensity for a bent lattice. The intensity curve is asymmetrical and gives a slow gradual drop on one side. If the three-dimensional nature of the lattice is taken into account, these streaks break up in a number of spots having the third Laue index l equal to $0, 1, 2, \cdots$. These coincide approximately with the arrangements of dots on parallel lines obtained by experiment. As a confirmation the computation of a pattern obtained from an accurately flat specimen by means of rotation of this specimen during exposure is quoted which is found to be equivalent to the pattern obtained from a bent specimen.

In an appendix a useful collection of asymptotic expressions for Bessel functions is given. *F. Kottler*

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AUTHOR INDEX

Abe, M. 121
Agnew, R. P. 132
Agostini, A. 114
Aigner, A. 115
Albert, A. A. 123
Archibald, R. C. 115
Baer, R. 122, 126
Balandin, A. A. 140
Barral Souto, J. 116
Barrett, W. 127
Bays, S.-Hsia, C.-C. 125
Bell, E. T. 113
Bernstein, S. N. 114
Bernolari, L. 136
Birkhoff, G. 120
Birkhoff, G. D. 133
Born, M. 115
Born, M.-Fuchs, K. 142
Born, M.-Fürth, R. 144
Born, M.-Hsiars, R. D. 144
Bosanquet, L. S. 131
Brauer, R.-Coxeter, H. S. M. ..125
Browne, E. T. 118
Burington, R. S. 118
Carathéodory, C. 138
Carlitz, L. 122
Chapman, S.-Ferraro, V. C. A. 141
Church, R. 120
Comessatti, A. 136
Coolidge, J. L. 113
Coppernicus, N. 115
Coxeter, H. S. M. See Brauer.
Dalzell, D. P. See Jeffreys.
van Dantzig, D. 139, 140
Datzeff, A. 142
Deruyts, J. 120
De Vries, H. 114
Dietzmann, A. P. 125
Dilworth, R. P. 120
Dingle, H. 115
Dooley, H. L. See Epstein.
Drinfeld, G. 127
Dubuque, P. E. 125
Emch, A. 137
Epstein, D. W.-Dooley, H. L. ..141
Ertel, H. 134
Etherington, I. M. H. 121
Ferrar, W. L. 115
Ferraro, V. C. A. See Chapman.
Finsler, P. 138
Fischer, O. F. 119
Fock, V. A.-Kolpinsky, V. A. ..144
Folley, K. W. 129
Frajese, A. 114
Franchietta, A. 137
Fuchs, K. 142, 143
Fuchs, K. See Born.
Fuchs, W. H. J. See Macintyre.
Fuchs-Rabinowitsch, D. J. 126
Fürth, R. See Born.
Furtwängler, F. 116
Geppert, H. 114
Gertschikoff, A. I. 121
Gheorghiu, G. T. 115
Gherardelli, G. 136
Gleyzal, A. 129
Godeaux, L. 138
Gomes, R. L. 118
Gontcharoff, V.-Kolmogoroff, A. 114
Graffi, D. 141
Griffin, H. 127
Guareschi, G. 132
Habich, W. 119
Hall, M. 122
Hall, P. 125, 126
Haupt, O.-Nöbeling, G.-Pauc, C. 130
Heins, A. E. 133
Hestenes, M. R.-McShane, E. J. 119
Hofmann, J. E. 113
Hofreiter, N. 115
Hornich, H. 131
Hornstein, M. 133
Hsia, C.-C. See Bays.
Huff, G. B. 136
Huxley, L. G. H. 140
Ivanov, V. 118
Jackson, F. H. 133
Jakovkin, M. V. 117
Jeans, J. 139
Jeffreys, H.-Dalzell, D. P. 134
Kakinuma, U. 144
Kametani, S. 131
Kantorovitch, L. V. 131
Kennedy, E. C. 116
Kharadse, A. 117
Kolmogoroff, A. See Gontcharoff.
Kolpinsky, V. A. See Fock.
Kramers, H. A. 140
Krasner, M. 123
Kulakoff, A. A. 124, 126
Kurosch, A. 127, 128
Lancaster, O. E. 132
Landherr, W. 118
Lange, H. O.-Neugebauer, O. ..114
Levi, F. W. 115, 128
Lichnerowicz, A.-Marrot, R. ..139
Lilley, S. 137
Loria, G. 116
McCrea, W. H. 142
MacLane, S. 123
McShane, E. J. See Hestenes.
Macintyre, A. J.-Fuchs, W. H. J. 117
Mahler, K. 119, 133
Malcev, A. 128
Markoff, A. 117
Markov, M. 142
Marrot, R. See Lichnerowicz.
Megyesi, I. 116
Meiman, N. N. 117
Miller, G. A. 124
Milne, E. A. 115
deMira Fernandes, A. 134
Misra, R. D. See Born.
Montel, P. 116
Moore, M. G. 132
Mori, S. 121
Müller, C. 114
Murray, F. J. 131
Nagamiya, T. 139
Narasimha Murti, V. 115
Neugebauer, O. See Lange.
Neumann, B. H. 121
v. Neumann, J.-Wigner, E. P. 127
Neumer, W. 127
Neville, E. H. 141
Nikol, F. 117
Nöbeling, G. See Haupt.
Obituaries 115
Oldenburger, R. 119
Oldenburger, R.-Porges, A. 119
Pannekoek, A. 114
Pauc, C. See Haupt.
Pauli, W. 144
Peters, T. 115
Petkantschin, B. 136
Phillips, H. B. 115
Piazzolla-Beloch 136
Piccard, S. 129
Pipes, L. A. 134, 141
Pompiu, G. 136
Porges, A. See Oldenburger.
Pospíšil, B. 131
Post, E. L. 128
Ptolemaeus, C. 115
Rachevsky, P. 135
Rashevsky, N. 138
Rees, D. 127
Robinson, G. de B. 135
Rosenfeld, L. 143
Saks, S. 129
Schneidmüller, V. I. 121
Schoenberg, I. J. 130
Scholz, A. 126
Schuchard, E. A.-Uehling, E. A. 140
Scott, D. B. 137
Sergescu, P. 116
Sigley, D. T. 125
Silberstein, L. 134
Smith, D. E. 114
Specht, W. 118, 126
Spiwak, G. V.-Zrebny, P. E. ..141
Steck, M. 135
Taussky, O.-Todd, J. 118
Teichmüller, O. 122
Thaer, C. 114
Thalberg, O. M. 136
Thompson, D'A. W. 115
Todd, J. See Taussky.
Toepken, H. 135
Tolman, R. C. 139
Tolstoff, G. 132
Tornier, E. 131
Tropfke, J. 114
Tschebotareff, N. G. 115, 117
Tschernikow, S. 126
Turkin, W. K. 125
Uehling, E. A. See Schuchard.
Vahlen, T. 135
Van der Lyn, G. 134
Vassiliou, P. 117
Vogel, K. 114
van der Waerden, B. L. 114, 120, 125
Wagner, K. W. 134
Ward, J. A. 122
Ward, M. 120
Weil, A. 123
Weiss, E. A. 114
Westerdijk, T. 141
de Wet, J. S. 144
Wigner, E. P. See v. Neumann.
Witt, E. 121
Woodcock, E. R. 137
Zariski, O. 124
Zassenhaus, H. 123
Zrebny, P. E. See Spiwak.

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